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Journal of Geometry and Physics 50 (2004) 273–302
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Hamiltonian operators and ℓ^*-coverings

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Received 14 April 2003; received in revised form 8 July 2003

Abstract

An efficient method to construct Hamiltonian structures for nonlinear evolution equations is described. It is based on the notions of variational Schouten bracket and ℓ^*-covering. The latter serves the role of the cotangent bundle in the category of nonlinear evolution PDEs. We first consider two illustrative examples (the KdV equation and the Boussinesq system) and reconstruct for them the known Hamiltonian structures by our methods. For the coupled KdV–mKdV system, a new Hamiltonian structure is found and its uniqueness (in the class of polynomial (x, t)-independent structures) is proved. We also construct a nonlocal Hamiltonian structure for this system and prove its compatibility with the local one.

MSC: 37K05; 35Q53

JGP SC: Dynamical systems

Keywords: Nonlinear evolution equations; Variational Schouten bracket; Hamiltonian structures; Recursion operators; Conservation laws; Coverings; The coupled KdV–mKdV system; The KdV equation; The Boussinesq equation

1. Introduction

We describe a method of constructing Hamiltonian structures for nonlinear evolution equations (or systems of such equations). The method is based on two concepts: the *variational Schouten bracket* and the *ℓ^*-covering* over a nonlinear PDE.

In Section 2, we expose some general facts concerning the geometry of super PDE. In Section 3, we construct the variational Schouten bracket on a super version of Kupershmidt's

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0393-0440/$ – see front matter
doi:10.1016/j.geomphys.2003.09.010

cotangent bundle to a bundle and, following [2], obtain an explicit formula for this bracket. In Section 4, simple computational formulas are deduced to check the Hamiltonianity of a bivector and compatibility of two Hamiltonian structures. Using the Schouten bracket, we define Hamiltonian evolution equations (including the cases when the Hamiltonian operator A may depend explicitly on time while the equation itself may not possess a Hamiltonian functional). This definition is equivalent to the operator equality

$$\ell_{\mathscr{E}} \circ A + A \circ \ell_{\mathscr{E}}^{*} = 0,$$

where $\ell_{\mathscr{E}}$ is the linearization of the equation and $\ell_{\mathscr{E}}^{*}$ is the adjoint operator. To solve this equation, we introduce the notion of $\ell_{\mathscr{E}}^{*}$-covering (which is a particular case of a more general construction introduced in Section 5) and show that to any operator A satisfying the above equation there corresponds a function s on the $\ell_{\mathscr{E}}^{*}$-covering such that $\tilde{\ell}_{\mathscr{E}}(s) = 0$, where $\tilde{\ell}_{\mathscr{E}}$ is the lifting of the linearization operator for $\mathscr{E}$ to the $\ell_{\mathscr{E}}^{*}$-covering. In other words, the operators we are interested in are identified with shadows of nonlocal symmetries (of a special type) in the $\ell_{\mathscr{E}}^{*}$-covering.

The reason to introduce the concept of ℓ^{*}-covering is twofold. First, as it was just indicated, it allows to reduce construction of Hamiltonian structures to computation of symmetries (with a subsequent check of additional conditions) for which a number of efficient software packages exists. Second, to our opinion, this point of view gives a new and fruitful insight into the theory of Hamiltonian structures for partial differential equations.

In Sections 6 and 7, these methods are applied to the known examples of the KdV equation and the Boussinesq system. In Section 8, we construct a Hamiltonian structure for the coupled KdV–mKdV system [5]. We also prove that this structure is unique in the class of (x, t)-independent polynomial structures. Nevertheless, extending the initial setting with certain nonlocal variables, we find another Hamiltonian operator that serves a Hamiltonian structure for ‘higher’ coupled KdV–mKdV equations. This structure is compatible with the local one. It is to be noted that the theory of nonlocal Hamiltonian structures is not sufficiently developed yet and needs additional research.

In the Appendix A, we briefly recall the construction of the recursion operator for the coupled KdV–mKdV system (obtained earlier in [5]) by which the above mentioned Hamiltonian structures are related to each other.

2. Generalities: jet bundles and differential equations

Let us formulate the main definitions and results we will use. For more details we refer to [1,2,8].

2.1. Jet bundles

Let $\pi : E \to M$ be a vector bundle over an n-dimensional base manifold M and $\pi_\infty : J^\infty(\pi) \to M$ be the infinite jet bundle of local sections of the bundle π.

In coordinate language, if $x_1, \ldots, x_n, u^1, \ldots, u^m$ are coordinates on E such that x_i are base coordinates and u^j are fiber ones, then $\pi_\infty : J^\infty(\pi) \to M$ is an infinite-dimensional vector bundle with fiber coordinates u^j_τ, where $\tau = i_1, \ldots, i_{|\tau|}$ is a symmetric multi-index.

Now, we generalize the definition of the jet bundle to the case of *superbundles*.

Definition 1. Let E be a supermanifold of superdimension $(n + m_0)|m_1$, and $\pi : E \to M$ be a vector bundle over an n-dimensional even manifold M. If π is split into the direct sum of two vector subbundles $\pi = \pi^0 \oplus \pi^1$ such that the fibers of π^0 are even and the fibers of π^1 are odd, then we say that π (along with the splitting) is a *superbundle*.

For a superbundle π, we define the *infinite jet superbundle* $\pi_\infty : J^\infty(\pi) \to M$ by setting:

$$(\pi_\infty)^0 = (\pi^0)_\infty, \qquad (\pi_\infty)^1 = (((\pi^1)^\Pi)_\infty)^\Pi,$$

where the superscript Π denotes the reversion of parity.

Denote by $\mathcal{F}(\pi)$ the superalgebra of smooth functions on $J^\infty(\pi)$.

Remark 1. By definition, we have

$$\mathcal{F}(\pi) = \mathcal{F}(\pi^0) \otimes_{C^\infty(M)} \Lambda^*(\mathcal{F}_{\mathrm{lin}}((\pi^1)^\Pi)),$$

where $\mathcal{F}_{\mathrm{lin}}(\cdot) \subset \mathcal{F}(\cdot)$ is the subspace of functions linear along fibers.

In what follows we shall use the term ‘bundle’ to mean ‘superbundle’.

2.2. The Cartan distribution

Consider a bundle $\pi : E \to M$ and define the $C^\infty(M)$-*supermodule* $\Gamma(\pi)$ of its ‘sections’ as follows. If π is even, then $\Gamma(\pi)$ is the module of sections of π. If π is a general superbundle, then we put $\Gamma(\pi) = \Gamma(\pi)^0 \oplus \Gamma(\pi)^1$, with $\Gamma(\pi)^0 = \Gamma(\pi^0)$ and $\Gamma(\pi)^1 = (\Gamma((\pi^1)^\Pi))^\Pi$.

Remark 2. Thus, in line with our definition of jets of superbundles, we define elements of $\Gamma(\pi)$ to be pairs of sections of π^0 and π^1.

Next, we note that every fiberwise linear function f on infinite jets $J^\infty(\pi)$ can be naturally identified with a linear differential operator $\nabla_f : \Gamma(\pi) \to C^\infty(M)$ and vice versa. Indeed, for even bundle π the correspondence is given in the relation

$$\nabla_f(s)(a) = f(j_\infty(s)(a)),$$

where $s \in \Gamma(\pi)$, $j_\infty(s)$ is the infinite jet of s, $a \in M$. The general case reduces to the even one, since $\mathcal{F}_{\mathrm{lin}}(\pi) = \mathcal{F}_{\mathrm{lin}}(\pi^0 \oplus (\pi^1)^\Pi)$.

Remark 3. The maps

$$j_\infty : \Gamma(\pi^0) \to \Gamma(\pi^0_\infty), \qquad j_\infty : \Gamma((\pi^1)^\Pi) \to \Gamma((\pi^1_\infty)^\Pi),$$

give rise to a map of supermodules $j_\infty : \Gamma(\pi) \to \Gamma(\pi_\infty)$.

The infinite jet bundle $\pi_\infty : J^\infty(\pi) \to M$ admits a natural flat connection such that the lift $\hat{X}$ of a vector field X on M is uniquely defined by the condition

$$\nabla_{\hat{X}(f)} = X \circ \nabla_f, \quad f \in \mathcal{F}_{\mathrm{lin}}(\pi).$$

In coordinates, the lift of $\partial/\partial x_i$ is the ith *total derivative*

$$\frac{\hat{\partial}}{\partial x_i} = D_i = \frac{\partial}{\partial x_i} + \sum_{j,\tau} u^j_{\tau i} \frac{\partial}{\partial u^j_\tau}.$$

Vector fields of the form $\hat{X}$ generate an n-dimensional distribution on $J^\infty(\pi)$ called the *Cartan distribution* and denoted by $\mathcal{C}(\pi)$. Obviously, the Cartan distribution is Frobenious in the sense that $[\mathcal{C}(\pi), \mathcal{C}(\pi)] \subset \mathcal{C}(\pi)$. In coordinate language, the Cartan distribution is spanned by the total derivatives.

2.3. Horizontal calculus and evolutionary fields

Let $\xi : B \to M$ be a vector bundle and $\pi^*_\infty(\xi) : B \times_M J^\infty(\pi) \to J^\infty(\pi)$ its pull-back along π_∞. The $C^\infty(J^\infty(\pi))$-supermodule $\Gamma(\pi^*_\infty(\xi))$ is defined as above: $\Gamma(\pi^*_\infty(\xi)) = \Gamma(\xi) \otimes_{C^\infty(M)} C^\infty(J^\infty(\pi))$, if ξ is even, and $\Gamma(\pi^*_\infty(\xi)) = \Gamma(\pi^*_\infty(\xi))^0 \oplus \Gamma(\pi^*_\infty(\xi))^1$, with $\Gamma(\pi^*_\infty(\xi))^0 = \Gamma(\pi^*_\infty(\xi)^0)$ and $\Gamma(\pi^*_\infty(\xi))^1 = (\Gamma((\pi^*_\infty(\xi)^1)^\Pi))^\Pi$ if ξ is a general superbundle.

Definition 2. A $C^\infty(J^\infty(\pi))$-(super)module P of the form $P = \Gamma(\pi^*_\infty(\xi))$ is said to be a *horizontal module*.

Example 1 (Horizontal forms). Let ξ be the qth exterior degree of the cotangent bundle to M. The corresponding horizontal module $\Gamma(\pi^*_\infty(\xi))$ is called the module of *horizontal forms* and is denoted by $\bar{\Lambda}^q(\pi)$. In coordinates, horizontal forms are generated by the forms $f\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_q}$, $f \in \mathcal{F}(\pi)$.

Definition 3. Let P and Q be $C^\infty(J^\infty(\pi))$-(super)modules. A map $\Delta : P \to Q$ is called *$\mathcal{C}$-differential operator* (or *horizontal operator*) if it can be written as a sum of compositions of $C^\infty(J^\infty(\pi))$-linear maps and vector fields of the form $\hat{X}$.

In coordinates, $\mathcal{C}$-differential operators are total derivatives operators.

Example 2 (The horizontal de Rham complex). We define the first horizontal de Rham differential $\bar{d} : \mathcal{F}(\pi) \to \bar{\Lambda}^1(\pi) = \Lambda^1(M) \otimes_{C^\infty(M)} \mathcal{F}(\pi)$ by the formula $\bar{d}(f)(X) = \hat{X}(f)$. In coordinates, we have $\bar{d}(f) = \sum_i D_i(f)\mathrm{d}x_i$.

The general horizontal differential $\bar{d} : \bar{\Lambda}^q(\pi) \to \bar{\Lambda}^{q+1}(\pi)$ is defined by the usual rules:

$$\bar{d} \circ \bar{d} = 0, \qquad \bar{d}(\omega_1 \wedge \omega_2) = \bar{d}\omega_1 \wedge \omega_2 + (-1)^q \omega_1 \wedge \bar{d}\omega_2, \qquad \omega_1 \in \bar{\Lambda}^q(\pi).$$

The differential $\bar{d}$ is a $\mathcal{C}$-differential operator.

The cohomology of the horizontal de Rham complex

$$0 \to \mathcal{F}(\pi) \xrightarrow{\bar{d}} \bar{\Lambda}^1(\pi) \xrightarrow{\bar{d}} \cdots \xrightarrow{\bar{d}} \bar{\Lambda}^n(\pi) \to 0$$

are called *horizontal cohomology* and denoted by $\bar{H}^q(\pi)$. From Vinogradov's 'one-line theorem' [12–14] it follows that $\bar{H}^q(\pi) = H^q(M)$ for $q \leq n-1$.

All $\mathcal{C}$-differential operators from P to Q form a $C^\infty(J^\infty(\pi))$-(super)module denoted by $\mathcal{C}\,\mathrm{Diff}(P, Q)$.

Clearly, if P and Q are horizontal, then so is $\mathcal{C}\,\mathrm{Diff}(P, Q)$.

Given a horizontal module P, let us define the *horizontal infinite jet bundle* $\pi_P : \bar{J}^\infty(P) \to J^\infty(\pi)$ as follows. If P is even, then the fiber of $\bar{J}^\infty(P)$ over $\theta \in J^\infty(\pi)$ consists of equivalence classes, denoted by $\bar{j}(p)(\theta)$, of elements $p \in P$. Two elements p_1 and p_2 are equivalent if their total derivatives of all orders coincide at θ. For a general horizontal supermodule P, we as always define $\pi_P^0 = \pi_{P^0}$ and $\pi_P^1 = \pi^{\Pi}_{(P^1)^\Pi}$. Correspondingly, $\Gamma(\pi_P) = \Gamma(\pi_{P^0}) \oplus \Gamma(\pi_{(P^1)^\Pi})^\Pi$.

Clearly, the horizontal jet bundle $\pi_P : \bar{J}^\infty(P) \to J^\infty(\pi)$, $P = \Gamma(\pi_\infty^*(\xi))$, is isomorphic to the pullback $\pi_\infty^*(\xi_\infty) : J^\infty(\xi) \times_M J^\infty(\pi) \to J^\infty(\pi)$ and, thus, $\Gamma(\pi_P)$ is a horizontal module.

Similarly to Remark 3, we have the natural operator $\bar{j}_\infty : P \to \bar{J}^\infty(P)$.

For every $\mathcal{C}$-differential operator $\Delta : P \to Q$ there exists a unique homomorphism of $C^\infty(J^\infty(\pi))$-supermodules $h_\Delta : \bar{J}^\infty(P) \to \bar{J}^\infty(Q)$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\Delta} & Q \\ \downarrow{\scriptstyle \bar{j}_\infty} & & \downarrow{\scriptstyle \bar{j}_\infty} \\ \bar{J}^\infty(P) & \xrightarrow{h_\Delta} & \bar{J}^\infty(Q) \end{array}$$

is commutative.

Let us recall the definition of adjoint operator. Consider $\Delta \in \mathcal{C}\,\mathrm{Diff}(P_1, P_2)$. The *adjoint operator* $\Delta^* \in \mathcal{C}\,\mathrm{Diff}(\hat{P}_2, \hat{P}_1)$, $\hat{P} = \mathrm{Hom}_{\mathcal{F}(\pi)}(P, \bar{\Lambda}^n(\pi))$, is uniquely defined by the equality[1]

$$\langle \hat{p}, \Delta(p) \rangle = (-1)^{\Delta \hat{p}} \langle \Delta^*(\hat{p}), p \rangle, \quad \hat{p} \in \hat{P}_2, \quad p \in P_1, \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing $\hat{P} \times P \to \bar{H}^n(\pi)$.

In coordinates, we have

$$\left\| \sum_\tau a_{ij}^\tau D_\tau \right\|^* = \left\| \sum_\tau (-1)^{|\tau|} D_\tau \circ a_{ij}^\tau \right\|^{\mathrm{st}},$$

where $a_{ij}^\tau \in \mathcal{F}(\pi)$, the superscript 'st' denotes the supertransposition, and $D_\tau = D_{i_1} \circ \cdots \circ D_{i_{|\tau|}}$ for $\tau = i_1, \ldots, i_{|\tau|}$.

[1] Here and below, symbols used at the exponents of (-1) stand for the corresponding parity.

Equivalently, adjoint operator can be defined using the following fact. Consider a horizontal module P and the natural complex

$$\begin{aligned}&0 \to \mathcal{C}\,\mathrm{Diff}(P, \mathcal{F}(\pi)) \to \mathcal{C}\,\mathrm{Diff}(P, \bar{\Lambda}^1(\pi)) \to \mathcal{C}\,\mathrm{Diff}(P, \bar{\Lambda}^2(\pi)) \to \cdots\\ &\quad \to \mathcal{C}\,\mathrm{Diff}(P, \bar{\Lambda}^{n-1}(\pi)) \to \mathcal{C}\,\mathrm{Diff}(P, \bar{\Lambda}^n(\pi)) \to 0\end{aligned}$$

with the differential $\Delta \mapsto \bar{d} \circ \Delta$. Denote its cohomology by $H^q(P)$. We have

$$\begin{cases} H^q(P) = 0 \\ H^n(P) = \hat{P} \end{cases} \quad \text{for } 0 < q < n. \tag{2}$$

Each $\mathcal{C}$-differential operator $\Delta : P \to Q$ gives rise to a cochain map between two such complexes. The corresponding map of the nth cohomology $\Delta^* : \hat{Q} \to \hat{P}$ is the adjoint operator. Note that the natural projection $\mu : \mathcal{C}\,\mathrm{Diff}(P, \bar{\Lambda}^n(\pi)) \to \hat{P}$ has the form $\Delta \mapsto \Delta^*(1)$.

Recall the most important properties of adjoint operators:

(1) Δ and Δ^* are of equal parity;
(2) $(\Delta_1 \circ \Delta_2)^* = (-1)^{\Delta_1 \Delta_2} \Delta_2^* \circ \Delta_1^*$;
(3) $\Delta^{**} = \Delta$ (here we identify $\hat{\hat{P}}$ and P).

A vector field Z on $J^\infty(\pi)$ is called *vertical* if $Z|_{C^\infty(M)} = 0$. For a horizontal module P a vertical field Z generates a natural action $Z : P \to P$, which in coordinates is the component-wise action.

A vertical vector field Z is said to be *evolutionary* if $[Z, \hat{X}] = 0$ for all vector fields X on M.

It is easy to see that evolutionary fields are uniquely determined by their restrictions to $\mathcal{F}_{\mathrm{lin}}(E)$, where E is the space of the bundle $\pi : E \to M$. Moreover, the map $Z \mapsto Z|_{\mathcal{F}_{\mathrm{lin}}(E)}$ is a bijection between the set of all evolutionary fields and $\mathrm{Hom}_{C^\infty(M)}(\mathcal{F}_{\mathrm{lin}}(E), \mathcal{F}(\pi))$. We identify $\mathrm{Hom}_{C^\infty(M)}(\mathcal{F}_{\mathrm{lin}}(E), \mathcal{F}(\pi))$ with the horizontal module $\Gamma(\pi_\infty^*(\pi))$ and denote it by $\varkappa(\pi)$.

In coordinate language, the evolutionary field that corresponds to a vector function $\varphi = (\varphi^1, \ldots, \varphi^m)$ has the form

$$\partial_\varphi = \sum_{j,\tau} D_\tau(\varphi^j) \frac{\partial}{\partial u_\tau^j}.$$

Let P be a horizontal module. The *linearization* of an element $F \in P$ is a $\mathcal{C}$-differential operator $\ell_F : \varkappa(\pi) \to P$ defined by the formula

$$\ell_F(\varphi) = (-1)^{F\varphi} \partial_\varphi(F).$$

Denote by the square brackets the horizontal cohomology class of a horizontal form. Since evolutionary fields commute with the horizontal differential, the cohomology class $[\partial_\varphi(\omega)]$ for $\omega \in \bar{\Lambda}^n(\pi)$ is well defined by $[\omega]$; denote it by $\partial_\varphi([\omega])$. By (1) we have

$$\partial_\varphi([\omega]) = [\partial_\varphi(\omega)] = (-1)^{\varphi\omega}[\ell_\omega(\varphi)] = \langle \varphi, \ell_\omega^*(1) \rangle = \langle \varphi, \mathcal{E}(\omega) \rangle,$$

where $\mathcal{E} : \bar{\Lambda}^n(\pi) \to \hat{\varkappa}(\pi)$, $\mathcal{E}(\omega) = \ell^*_\omega(1)$, is the *Euler operator*, which takes Lagrangians to the corresponding Euler–Lagrange equations. Of course, the value $\mathcal{E}(\omega)$ is completely determined by the cohomology class $[\omega]$.

In coordinates, $\mathcal{E}(L\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n) = (\delta L/\delta u^1, \ldots, \delta L/\delta u^m)$, where $\delta L/\delta u^j = \sum_\tau (-1)^{|\tau|} D_\tau(\partial L/\partial u^j_\tau)$.

Remark 4. From Vinogradov's 'one-line theorem' [12–14] it follows that:

(1) $\ker \mathcal{E}/\bar{d}(\bar{\Lambda}^{n-1}(\pi)) = H^n(M)$;
(2) $\psi \in \operatorname{im} \mathcal{E}$ if and only if $\ell^*_\psi = \ell_\psi$.

2.4. *Differential equations*

Again, consider an element F of a horizontal module P. The locus

$$\mathscr{E}^\infty = \{\bar{\jmath}_\infty(F) = 0\} \subset J^\infty(\pi)$$

is called *differential equation* defined by F. We assume that the natural map $\mathscr{E}^\infty \to M$ is a subbundle of the bundle $\pi : J^\infty(\pi) \to M$. The restriction of the Cartan distribution to $\mathscr{E}^\infty$ is denoted by $\mathcal{C}(\mathscr{E})$. Clearly, $\dim \mathcal{C}(\mathscr{E}) = \dim(\mathcal{C}) = n$.

Example 3 (Evolution equations). Consider the bundle $\bar{\pi} : E \times \mathbb{R} \to M \times \mathbb{R}$. Denote the coordinate along $\mathbb{R}$ by t. Then $\partial_\Phi = D_t - (\partial/\partial t)$ is a canonical evolutionary field on $J^\infty(\bar{\pi})$. In coordinates, $\Phi = (u^1_t, \ldots, u^m_t)$. Let $\partial_{\varphi(t)}$ be a family of evolutionary fields on $J^\infty(\pi)$. The equation $\mathscr{E}^\infty \subset J^\infty(\bar{\pi})$ given by the element $F = \Phi - \varphi(t) \in \varkappa(\bar{\pi})$ is called *evolution equation*. In coordinates, it has the form $u_t = \varphi(t)$. Note that $\mathscr{E}^\infty = J^\infty(\pi) \times \mathbb{R}$, with the Cartan distribution generated by that on $J^\infty(\pi)$ and the field $D_t = (\partial/\partial t) + \partial_{\varphi(t)}$.

The restriction of the linearization ℓ_F to the equation $\mathscr{E}^\infty$ is called the *linearization of* $\mathscr{E}^\infty$ and is denoted by $\ell_{\mathscr{E}} : \varkappa \to P$, where $\varkappa = \varkappa(\pi)|_{\mathscr{E}^\infty}$.

An evolutionary field tangent to $\mathscr{E}^\infty$ is said to be a *symmetry* of the equation. Obviously, ∂_φ is a symmetry if and only if $\ell_{\mathscr{E}}(\varphi) = 0$, $\varphi \in \varkappa$.

The horizontal de Rham complex on $J^\infty(\pi)$ can be restricted to $\mathscr{E}^\infty$. Its cohomology are called *horizontal cohomology of equation* $\mathscr{E}^\infty$ and denoted by $\bar{H}^q(\mathscr{E})$. Elements of $\bar{H}^{n-1}(\mathscr{E})/H^{n-1}(M)$ are *conservation laws* of $\mathscr{E}^\infty$. If the equation at hand satisfies the conditions of Vinogradov's 'two-line theorem' [12–14] then there is an inclusion $i : \bar{H}^{n-1}(\mathscr{E})/H^{n-1}(M) \to \ker \ell^*_{\mathscr{E}}$. The element $i(\eta) \in \ker \ell^*_{\mathscr{E}} \subset \hat{P}$ that corresponds to conservation law η is called its *generating function*.

In particular, evolution equations satisfy the conditions of the two-line theorem. In this case, $i(\eta) = \mathcal{E}(\eta_0)$, where $\eta = \eta_0 + \eta_1 \wedge \mathrm{d}t$, $\eta_0 \in \bar{\Lambda}^{n-1}(\mathscr{E})$, $\eta_1 \in \bar{\Lambda}^{n-2}(\mathscr{E})$. Thus, to find conservation laws of an evolution equation one has to solve the equation $\ell^*_{\mathscr{E}}(\psi) = 0$ and choose those solutions ψ that fulfill the condition $\ell_\psi = \ell^*_\psi$.

Let $\mathscr{E}^\infty$ and $\tilde{\mathscr{E}}^\infty$ be two differential equations. A surjective map $\tau : \tilde{\mathscr{E}} \to \mathscr{E}^\infty$ is called *covering* if it preserves the Cartan distribution.

Example 4. A horizontal jet bundle $\pi_P : \bar{J}^\infty(P) \to J^\infty(\pi)$ is a covering. A generalization of this example will be discussed in Section 5.

Example 5. Let $\mathscr{E}^\infty$ be given by an element F. Consider equation $\widetilde{\mathscr{E}}^\infty$

$$F = 0, \frac{\partial r}{\partial x_1} = g_1, \frac{\partial r}{\partial x_1} = g_1, \dots, \frac{\partial r}{\partial x_n} = g_n, \tag{3}$$

where $g_1, \dots, g_n$ are functions of $x_1, \dots, x_n$ and u^j_σ. If the compatibility condition

$$D_i(g_j) = D_j(g_i) \text{ on } \mathscr{E}^\infty \Leftrightarrow \bar{d}\eta|_{\mathscr{E}^\infty} = 0 \quad \eta = \sum_i g_i \mathrm{d}x_i \in \bar{\Lambda}^1(\mathscr{E}) \tag{4}$$

holds true, then the natural projection $\tau : \widetilde{\mathscr{E}}^\infty \to \mathscr{E}^\infty$ is epimorphic. Obviously, τ preserves the Cartan distribution, so that it is a covering. Thus, each closed horizontal one-form gives rise to a covering of the form (3). In particular, when $n = 2$, condition (4) means that η represents a conservation law of $\mathscr{E}^\infty$. The new dependent variable r is called *nonlocal variable*.

In a similar way, we can define a covering over $\widetilde{\mathscr{E}}^\infty$ corresponding to a closed one-form on this equation, etc. In this manner we construct particular coverings in Sections 6–8.

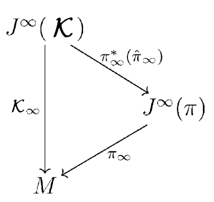
Clearly, each $\mathcal{C}$-differential operator Δ on $\mathscr{E}^\infty$ can be lifted to a $\mathcal{C}$-differential operator $\tilde{\Delta}$ on $\widetilde{\mathscr{E}}^\infty$. In particular, we have the operator $\tilde{\ell}_{\mathscr{E}}$ on $\widetilde{\mathscr{E}}^\infty$. A symmetry of $\widetilde{\mathscr{E}}^\infty$ is called a *nonlocal symmetry* of $\mathscr{E}^\infty$ in the covering under consideration. Solutions of the equation $\tilde{\ell}_{\mathscr{E}}(\varphi) = 0$ are called *shadows of nonlocal symmetries* of $\mathscr{E}^\infty$ in this covering. In a similar way, since the horizontal de Rham differential is a $\mathcal{C}$-differential operator, we can lift the horizontal de Rham complex to $\widetilde{\mathscr{E}}^\infty$ and construct the theory of *nonlocal conservation laws* in our covering. Solutions of the equation $\tilde{\ell}^*_{\mathscr{E}}(\psi) = 0$ are called *nonlocal generating functions*.

3. Variational Schouten bracket

We start with a super version of Kupershmidt's *cotangent bundle to a vector bundle* [9].

For a vector bundle $\pi : E \to M$, $\dim M = n$, we consider the bundle $\hat{\pi} : \hat{E} = E^* \otimes_M \Lambda^n(T^*M) \to M$, where $E^* \to M$ is the dual bundle to $E \to M$, and the superbundle $\mathcal{K} : \mathcal{K}^0 = \pi$ (even subbundle), $\mathcal{K}^1 = \hat{\pi}$ (odd subbundle).

The superbundle $\mathcal{K}_\infty : J^\infty(\mathcal{K}) \to M$



is called the *cotangent bundle* of the bundle π. It is clear that

$$J^\infty(\mathcal{K}) = \bar{J}^\infty(\Gamma(\pi^*_\infty(\hat{\pi}_\infty))).$$

Denote by p^j, $j = 1, \dots, m$, the fiber coordinates in $\hat{E}$ dual to u^j with respect to a volume form on M (they are sometimes called ‘antifields’). Then x_i, u^j_τ, p^j_τ will be the coordinates in $J^\infty(\mathcal{K})$, with x_i, u^j_τ being even and p^j_τ being odd.

It is clear that $\varkappa(\mathcal{K})^0 = \varkappa_{\mathcal{K}}$ and $\varkappa(\mathcal{K})^1 = \hat{\varkappa}^{\Pi}_{\mathcal{K}}$, where $\varkappa_{\mathcal{K}} = \Gamma(\mathcal{K}^*(\pi))$.

Define the *variational Schouten bracket* (*antibracket*) on the space $\bar{H}^n(\mathcal{K})$ by putting

$$[\![F, H]\!] = \langle \mathcal{E}(H), \alpha(\mathcal{E}(F)) \rangle, \qquad F, H \in \bar{H}^n(\mathcal{K}), \tag{5}$$

where $\mathcal{E}$ is the Euler operator and the operator $\alpha : \hat{\varkappa}(\mathcal{K}) \to \varkappa(\mathcal{K})$ acts according to the formula $\alpha(\psi, \varphi) = (\varphi, -\psi)$ for $\varphi \in \varkappa_{\mathcal{K}}$ and $\psi \in \hat{\varkappa}_{\mathcal{K}}$. In coordinates, we get

$$[\![F, H]\!] = \sum_j \left[\frac{\delta H}{\delta u^j} \frac{\delta F}{\delta p^j} - (-1)^{(F+1)(H+1)} \frac{\delta F}{\delta u^j} \frac{\delta H}{\delta p^j} \right].$$

It is readily seen that the variational Schouten bracket defines a Lie superalgebra structure on $\bar{H}^n(\mathcal{K})$:

$$\begin{aligned} &[\![H, F]\!] = -(-1)^{(F+1)(H+1)} [\![F, H]\!], \\ &(-1)^{(F+1)(G+1)} [\![[\![F, G]\!], H]\!] + (-1)^{(G+1)(H+1)} [\![[\![G, H]\!], F]\!] + \\ &\qquad (-1)^{(H+1)(F+1)} [\![[\![H, F]\!], G]\!] = 0. \end{aligned}$$

Remark 5. A different concept of the Schouten bracket (acting on a different space) the reader can find in [15, p. 226].

Denote by $\mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k)}(P, Q)$ the module of k-linear skew-symmetric $\mathcal{C}$-differential operators $P \times \cdots \times P \to Q$. The subset $\mathcal{C}\mathrm{Diff}^{\mathrm{sk\text{-}ad}}_{(k)}(P, \hat{P}) \subset \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k)}(P, \hat{P})$ consists of skew-adjoint in each argument operators.

Let us define multiplication

$$\mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k)}(P, \mathcal{F}(\pi)) \times \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(l)}(P, \mathcal{F}(\pi)) \to \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k+l)}(P, \mathcal{F}(\pi)),$$

by setting

$$(\Delta_1 \Delta_2)(p_1, \dots, p_{k+l}) = \sum_{\sigma \in S^k_{k+l}} (-1)^\sigma \Delta_1(p_{\sigma(1,k)}) \Delta_2(p_{\sigma(k+1,k+l)}),$$

where $\Delta_1 \in \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(k)}(P, \mathcal{F}(\pi))$, $\Delta_2 \in \mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(l)}(P, \mathcal{F}(\pi))$, $S^i_n \subset S_n$ is the set of all $(i, n-i)$-unshuffles [10], i.e., all permutations $\sigma \in S_n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(i)$ and $\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n)$, $(-1)^\sigma$ is the sign of permutation σ, and $p_{\sigma(k_1,k_2)}$ stands for $p_{\sigma(k_1)}, \dots, p_{\sigma(k_2)}$.

Next, since by definition elements of $\mathcal{F}(\hat{\pi})$ are identified with differential operators from $\Gamma(\hat{\pi})$ to $C^\infty(M)$, we have the natural inclusion $\mathcal{C}\mathrm{Diff}(\hat{\varkappa}(\pi), \mathcal{F}(\pi)) \to \mathcal{F}(\mathcal{K})$, which uniquely prolongs to the isomorphism of algebras

$$\mathcal{C}\mathrm{Diff}^{\mathrm{skew}}_{(*)}(\hat{\varkappa}(\pi), \mathcal{F}(\pi)) \to \mathcal{F}(\mathcal{K}).$$

Using (2) we can show in a standard way that

$$\bar{H}^n(\mathcal{K}) = \mathcal{C}\,\mathrm{Diff}^{\text{sk-ad}}_{(*)}(\hat{\varkappa}(\pi), \varkappa(\pi)) \oplus \bar{H}^n(\pi). \tag{6}$$

Below, we use the shorthand notation $\varkappa = \varkappa(\pi)$.

Now, following [2], we want to compute the variational Schouten bracket in terms of skew-adjoint $\mathcal{C}$-differential operators.

To this end, note that from the definition of the Euler operator it follows that its restriction:

$$\mathcal{E}|_{\mathcal{C}\,\mathrm{Diff}^{\text{skew}}_{(k)}(\hat{\varkappa}, \bar{\Lambda}^n(\pi))} : \mathcal{C}\,\mathrm{Diff}^{\text{skew}}_{(k)}(\hat{\varkappa}, \bar{\Lambda}^n(\pi)) \to \mathcal{C}\,\mathrm{Diff}^{\text{skew}}_{(k)}(\hat{\varkappa}, \hat{\varkappa}) \oplus \mathcal{C}\,\mathrm{Diff}^{\text{skew}}_{(k-1)}(\hat{\varkappa}, \varkappa),$$

has the form $\mathcal{E}|_{\mathcal{C}\,\mathrm{Diff}^{\text{skew}}_{(k)}(\hat{\varkappa}, \bar{\Lambda}^n(\pi))}(\Delta) = (\eta(\Delta), (-1)^{k-1}\mu(\Delta))$, where

$$\eta(\Delta)(\psi_1, \dots, \psi_k) = \ell^*_{\Delta,\psi_1,\dots,\psi_k}(1),$$
$$\mu(\Delta)(\psi_1, \dots, \psi_{k-1}) = (\Delta(\psi_1, \dots, \psi_{k-1}))^*(1),$$

$\psi_i \in \hat{\varkappa}$, $\ell_{\Delta,\psi_1,\dots,\psi_k}(\varphi) = \partial_\varphi(\Delta)(\psi_1, \dots, \psi_k)$.

In coordinates, $\eta = (\delta/\delta u^1, \dots, \delta/\delta u^m)$ and $\mu = (-1)^{k-1}(\delta/\delta p^1, \dots, \delta/\delta p^m)$.

We can rewrite η in the following form:

$$\eta(\Delta) = \tilde{\eta}(\mu(\Delta)), \qquad \Delta \in \mathcal{C}\,\mathrm{Diff}^{\text{skew}}_{(k)}(\hat{\varkappa}, \bar{\Lambda}^n(\pi)),$$

where

$$\tilde{\eta}(\Box)(\psi_1, \dots, \psi_k) = \ell^*_{\Box,\psi_1,\dots,\psi_{k-1}}(\psi_k), \qquad \Box \in \mathcal{C}\,\mathrm{Diff}^{\text{sk-ad}}_{(k-1)}(\hat{\varkappa}, \varkappa).$$

Indeed, take the equality

$$[\Delta(\psi_1, \dots, \psi_k)] = \langle \Box(\psi_1, \dots, \psi_{k-1}), \psi_k \rangle, \qquad \Box = \mu(\Delta),$$

and apply ∂_φ to both sides. This yields

$$[\partial_\varphi(\Delta)(\psi_1, \dots, \psi_k)] = \langle \partial_\varphi(\Box)(\psi_1, \dots, \psi_{k-1}), \psi_k \rangle,$$

and so

$$\langle \varphi, \ell^*_{\Delta,\psi_1,\dots,\psi_k}(1) \rangle = \langle \varphi, \ell^*_{\Box,\psi_1,\dots,\psi_{k-1}}(\psi_k) \rangle.$$

Thus, for the variational Schouten bracket of two operators $A \in \mathcal{C}\,\mathrm{Diff}^{\text{sk-ad}}_{(k-1)}(\hat{\varkappa}, \varkappa)$ and $B \in \mathcal{C}\,\mathrm{Diff}^{\text{sk-ad}}_{(l-1)}(\hat{\varkappa}, \varkappa)$ we have

$$\begin{aligned}
&\langle [\![A, B]\!](\psi_1, \dots, \psi_{k+l-2}), \psi_{k+l-1} \rangle \\
&\quad = [((-1)^{k-1}\tilde{\eta}(B)A - (-1)^{k(l-1)}\tilde{\eta}(A)B)(\psi_1, \dots, \psi_{k+l-1})] \\
&\quad = (-1)^{k-1} \sum_{\sigma \in S^l_{k+l-1}} (-1)^\sigma \langle \ell^*_{B,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}), A(\psi_{\sigma(l+1,k+l-1)}) \rangle - (-1)^{k(l-1)} \\
&\qquad \times \sum_{\sigma \in S^k_{k+l-1}} (-1)^\sigma \langle \ell^*_{A,\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}), B(\psi_{\sigma(k+1,k+l-1)}) \rangle.
\end{aligned}$$

Here and below we assume that $S^i_n = \varnothing$ if $i < 0$ or $i > n$.

Let us split the sums obtained into two parts depending on whether $\sigma(k+l-1) = k+l-1$ or not:

$$\begin{aligned}
&\langle [\![A, B]\!](\psi_1, \dots, \psi_{k+l-2}), \psi_{k+l-1}\rangle \\
&\quad = (-1)^{k-1} \sum_{\sigma\in S^l_{k+l-2}} (-1)^\sigma \langle \ell^*_{B,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}), A(\psi_{\sigma(l+1,k+l-2)}, \psi_{k+l-1})\rangle \\
&\qquad + \sum_{\sigma\in S^{l-1}_{k+l-2}} (-1)^\sigma \langle \ell^*_{B,\psi_{\sigma(1,l-1)}}(\psi_{k+l-1}), A(\psi_{\sigma(l,k+l-2)})\rangle - (-1)^{k(l-1)} \\
&\qquad \times \sum_{\sigma\in S^{k}_{k+l-2}} (-1)^\sigma \langle \ell^*_{A,\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}), B(\psi_{\sigma(k+1,k+l-2)}, \psi_{k+l-1})\rangle - (-1)^{(k-1)(l-1)} \\
&\qquad \times \sum_{\sigma\in S^{k-1}_{k+l-2}} (-1)^\sigma \langle \ell^*_{A,\psi_{\sigma(1,k-1)}}(\psi_{k+l-1}), B(\psi_{\sigma(k,k+l-2)})\rangle.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&[\![A, B]\!](\psi_1, \dots, \psi_{k+l-2}) \\
&\quad = \sum_{\sigma\in S^{l-1}_{k+l-2}} (-1)^\sigma \ell_{B,\psi_{\sigma(1,l-1)}}(A(\psi_{\sigma(l,k+l-2)})) - (-1)^{(k-1)(l-1)} \\
&\qquad \times \sum_{\sigma\in S^{k}_{k+l-2}} (-1)^\sigma B(\ell^*_{A,\psi_{\sigma(1,k-1)}}(\psi_{\sigma(k)}), \psi_{\sigma(k+1,k+l-2)}) - (-1)^{(k-1)(l-1)} \\
&\qquad \times \sum_{\sigma\in S^{k-1}_{k+l-2}} (-1)^\sigma \ell_{A,\psi_{\sigma(1,k-1)}}(B(\psi_{\sigma(k,k+l-2)})) \\
&\qquad + \sum_{\sigma\in S^{l}_{k+l-2}} (-1)^\sigma A(\ell^*_{B,\psi_{\sigma(1,l-1)}}(\psi_{\sigma(l)}), \psi_{\sigma(l+1,k+l-2)}). \qquad (7)
\end{aligned}$$

From the definition it immediately follows that:

$$[\![A, \omega]\!](\psi_1, \dots, \psi_{k-2}) = A(\mathcal{E}(\omega), \psi_1, \dots, \psi_{k-2})$$

for $\omega \in \bar{H}^n(\pi)$; in particular, $[\![\varphi, \omega]\!] = \langle \varphi, \mathcal{E}(\omega)\rangle = \partial_\varphi(\omega)$.

4. Hamiltonian evolution equations

An operator $A \in \mathcal{C}\mathrm{Diff}(\hat{\varkappa}, \varkappa)$ is called *Hamiltonian* if $[\![A, A]\!] = 0$. As in the classical Hamiltonian formalism, a Hamiltonian operator defines a Lie algebra structure on $\bar{H}^n(\pi)$ via the *Poisson bracket*

$$\{\omega_1, \omega_2\}_A = \langle A(\mathcal{E}(\omega_1)), \mathcal{E}(\omega_2)\rangle.$$

Remark 6. Hamiltonian operators are uniquely determined by the corresponding Poisson brackets.

Remark 7. A Hamiltonian operator A gives rise to a complex

$$0 \to \bar{H}^n(\pi) \stackrel{\partial_A}{\to} \varkappa \stackrel{\partial_A}{\to} \mathcal{C}\,\mathrm{Diff}(\hat{\varkappa}, \varkappa) \stackrel{\partial_A}{\to} \mathcal{C}\,\mathrm{Diff}_2^{\text{sk-ad}}(\hat{\varkappa}, \varkappa) \stackrel{\partial_A}{\to} \cdots, \tag{8}$$

where $\partial_A(\Delta) = [\![A, \Delta]\!]$, called the *Hamiltonian complex*.

Formula (7) yields a well-known criterion for checking a skew-adjoint operator to be Hamiltonian (see, e.g., [1,8]):

$$[\![A, A]\!](\psi_1, \psi_2) = -\ell_{A,\psi_1}(A(\psi_2)) + \ell_{A,\psi_2}(A(\psi_1)) - A(\ell^*_{A,\psi_1}(\psi_2)) = 0.$$

Another practical way to check the Hamiltonian property of an operator is to use formula (5). In coordinates, it gives:

$$\sum_j \frac{\delta W_A}{\delta u^j}\frac{\delta W_A}{\delta p^j} = 0 \ \text{ modulo total derivatives or } \ \mathcal{E}\left(\sum_j \frac{\delta W_A}{\delta u^j}\frac{\delta W_A}{\delta p^j}\right) = 0, \tag{9}$$

where $W_A \in \bar{H}^n(\mathcal{K})$ is the element that corresponds to the operator $2A$ under the isomorphism (6). In coordinates, the element W_A for an operator $\sum_\tau a^{ij}_\tau D_\tau$ has the form $W_A = \sum_{\tau ij} a^{ij}_\tau p^j_\tau p^i$.

The condition for two Hamiltonian operators A and B to be a Hamiltonian pair, i.e., $[\![A, B]\!] = 0$, is

$$\sum_j \mathcal{E}\left(\frac{\delta W_A}{\delta u^j}\frac{\delta W_B}{\delta p^j} + \frac{\delta W_B}{\delta u^j}\frac{\delta W_A}{\delta p^j}\right) = 0.$$

Note that the skew-adjointness in terms of W_A amounts to the equality

$$\sum_j \frac{\delta W_A}{\delta p^j} p^j = -2W_A. \tag{10}$$

Let A be a Hamiltonian operator. Evolution equation $u_t = f$ is said to be *Hamiltonian* with respect to A if

$$A_t - [\![A, f]\!] = 0, \tag{11}$$

where $A_t = \partial A/\partial t$ (both A and f can depend on the parameter t).

If A does not depend on t, then for each $H \in \bar{H}^n(\pi)$ the evolution equation $u_t = A(\mathcal{E}(H))$ is a Hamiltonian evolution equation. The element $H \in \bar{H}^n(\pi)$ is called the *Hamiltonian*. Notice that in this case condition (11) means that f is a one-cocycle in the Hamiltonian complex (8). A Hamiltonian H exists if and only if f is a coboundary.

If a Hamiltonian H exists and does not depend on t, then we have

$$D_t(H) = \partial_f(H) = \partial_{A(\mathcal{E}(H))}(H) = \{H, H\}_A = 0.$$

Thus, there exists a conservation law given by $\eta_0 + \eta_1 \wedge \mathrm{d}t$, $\eta_0 \in \bar{\Lambda}^n(\pi)$, $\eta_1 \in \bar{\Lambda}^{n-1}(\pi)$, such that $[\eta_0] = H \in \bar{H}^n(\pi)$, where $[\eta_0]$ is the cohomology class of η_0 in $\bar{H}^n(\pi)$. In other words, the generating function of this conservation law equals $\mathcal{E}(H)$. This conservation law is called the *conservation law of energy*.

Theorem 1. *Let $\mathscr{E}^\infty$ be an evolution equation $u_t = f$ which is Hamiltonian with respect to a Hamiltonian operator A. Then we have*

$$\ell_{\mathscr{E}} \circ A + A \circ \ell^*_{\mathscr{E}} = 0. \tag{12}$$

Proof. By (7)

$$(A_t - [\![A, f]\!])(\psi) = A_t(\psi) + \ell_{A,\psi}(f) - A(\ell^*_f(\psi)) - \ell_f(A(\psi)),$$

thus

$$A_t - [\![A, f]\!] = A_t + \partial_f(A) - A \circ \ell^*_f - \ell_f \circ A.$$

Hence, $((\partial/\partial t) + \partial_f - \ell_f) \circ A - A \circ ((\partial/\partial t) + \partial_f + \ell^*_f) = 0$. It remains to note that $\ell_{\mathscr{E}} = D_t - \ell_f = (\partial/\partial t) + \partial_f - \ell_f$. □

Remark 8. For equations possessing a Hamiltonian, relation (12) can be found elsewhere (see, e.g. [11]).

We call solutions of (12) *variational bivectors on the equation* under consideration; Hamiltonian operators that make a given equation Hamiltonian are, thus, special variational bivectors on the equation. Obviously, variational bivectors (and, in particular, Hamiltonian operators) take generating functions of conservation laws of the equation at hand to symmetries of this equation.

Proposition 1. *Let $\mathscr{E}^\infty$ be an evolution equation $u_t = f$. If two operators $A, A' \in \mathcal{C}\,\mathrm{Diff}(\hat{\varkappa}, \varkappa)$ satisfy the equation*

$$\ell_{\mathscr{E}} \circ A + A' \circ \ell^*_{\mathscr{E}} = 0, \tag{13}$$

then $A' = A$.

Proof. Rewrite (13) in the form

$$(D_t - \ell_f) \circ A - A' \circ (D_t + \ell^*_f) = 0.$$

Commute the right-hand side of this equality with the operator of multiplication by t. This gives $A' = A$. □

5. *Δ*-coverings

In this section, we describe a construction, that reduces solution of Eq. (12) to finding shadows of nonlocal symmetries in a special covering over $\mathscr{E}$ (the $\ell^*_{\mathscr{E}}$-covering).

Let $\mathcal{E}^\infty$ be a differential equation, and $\Delta : P \to Q$ be a $\mathcal{C}$-differential operator between two horizontal modules P and Q over $\mathcal{E}^\infty$. Consider the homomorphism $h_\Delta : \bar{J}^\infty(P) \to \bar{J}^\infty(Q)$ that corresponds to Δ. If $K_\Delta = \ker h_\Delta \subset \bar{J}^\infty(P)$ is a subbundle of $\bar{J}^\infty(P)$, then $k_\Delta = \pi_P|_{K_\Delta} : K_\Delta \to \mathcal{E}^\infty$ is a covering. We call it Δ-*covering*.

In terms of local coordinates, if $\Delta = \left\| \sum_\tau a_\tau^{ij} D_\tau \right\|$ and w^j are fiber coordinates of α, where α is such that $P = \Gamma(\alpha)$, then Δ-covering is defined by the equations

$$\sum_{\tau j} a_\tau^{ij} w_\tau^j = 0. \tag{14}$$

We can think of fibers of Δ-covering as even or odd. Here we prefer the latter viewpoint, so that k_Δ is a superbundle.

Δ-coverings are useful mainly due to the following obvious fact.

Proposition 2. *Let $R = \Gamma(\gamma)$ be a horizontal module over $\mathcal{E}^\infty$. Then there is an isomorphism*

$$\Gamma_{\mathrm{lin}}(k_\Delta^*(\gamma)) = \mathcal{C}\,\mathrm{Diff}(P, R)/\{V \in \mathcal{C}\,\mathrm{Diff}(P, R) | V = \square \circ \Delta, \square \in \mathcal{C}\,\mathrm{Diff}(Q, R)\},$$

where Γ_{lin} denotes space of fiberwise linear sections.

Proof. The isomorphism takes $s \in \Gamma_{\mathrm{lin}}(k_\Delta^*(\gamma))$ to the equivalence class of the operator $V_s : P \to R$ given by the formula

$$V_s(p) = \tilde{s} \circ \bar{j}_\infty(p),$$

where $\tilde{s}$ is an extension of s to $\bar{J}^\infty(P)$, $p \in P$. □

In coordinate language, D_τ at the jth component of the operator goes to w_τ^j.

Now suppose that we are given a $\mathcal{C}$-differential operator $\nabla : R \to R'$ over $\mathcal{E}^\infty$. Let us lift it on K_Δ and consider $\ker \nabla$. In view of Proposition 2, we can identify fiberwise linear elements of $\ker \nabla$ with solutions $V \in \mathcal{C}\,\mathrm{Diff}(P, R)/\{\square \circ \Delta\}$ of the equation

$$\nabla \circ V = V' \circ \Delta. \tag{15}$$

Thus, (15) *amounts to the equation* $\nabla(s) = 0$ *on the Δ-covering.*

In particular, Eq. (13) is equivalent to the equation $\ell_{\mathcal{E}}(\varphi) = 0$ on the $\ell_{\mathcal{E}}^*$-covering, where φ is fiberwise linear vector function. Note that in this case the $\ell_{\mathcal{E}}^*$-covering can be identified with the cotangent bundle $J^\infty(\mathcal{K}) \times \mathbb{R}$. Under this identification, the Cartan planes on $K_{\ell_{\mathcal{E}}^*}$ are spans of the Cartan planes on $J^\infty(\mathcal{K})$ and $D_t = (\partial/\partial t) + \mathcal{Э}_{\tilde{f}}$, where $\tilde{f} = (f, \ell_f^*(w))$, if the equation at hand is $u_t = f$. Moreover, the equivalence classes of operators from Proposition 2 are in one-to-one correspondence with $\mathcal{C}$-differential operators on $J^\infty(\pi)$.

Remark 9. From the above said, we see that Hamiltonian operators are shadows of nonlocal symmetries in the $\ell_{\mathcal{E}}^*$-covering.

Remark 10. Solutions $V \in \mathcal{C}\,\mathrm{Diff}(P, R)/\{\square \circ \Delta\}$ of Eq. (15) can be found straightforwardly and the computations will be essentially the same as when one solves the equation $\nabla(s) =$

0 on the Δ-covering. Nevertheless, in our computations we prefer the second approach, because in the case $\nabla = \ell_{\mathscr{E}}$ it reduces the problem to finding shadows of nonlocal symmetries (see above) for which efficient software exists.

6. The Korteweg–de Vries equation

Here we show how the above introduced techniques work with a simple and well-known example of the KdV equation

$$u_t = u_{xxx} + uu_x. \tag{16}$$

Local coordinates in $\mathscr{E}^\infty$ are

$$x, t, u = u_0, \dots, u_k, \dots,$$

where $u_k = \partial^k u/\partial x^k$ (similar notation is used in the subsequent sections as well). In these coordinates, the total derivatives are

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} u_{k+1}\frac{\partial}{\partial u_k}, \qquad D_t = \frac{\partial}{\partial t} + \sum_{k\geq 0} D_x^k(u_3 + uu_1)\frac{\partial}{\partial u_k}.$$

6.1. *The $\ell^*_{\mathscr{E}}$-covering*

The linearization operator for (16) is

$$\ell_{\mathscr{E}} = D_t - D_x^3 - uD_x - u_1,$$

while the adjoint is expressed by the formula

$$\ell^*_{\mathscr{E}} = -D_t + D_x^3 + uD_x.$$

Following the general scheme, we construct the $\ell^*_{\mathscr{E}}$-covering by introducing the odd variables $p = p_0$, $p_k = D_x^k(p)$ that satisfy the equation

$$p_t = p_3 + up_1.$$

6.2. *Solving the defining equation*

Let us now extend the total derivatives up to the total derivatives on the $\ell^*_{\mathscr{E}}$-covering

$$\tilde{D}_x = D_x + \sum_{k\geq 0} p_{k+1}\frac{\partial}{\partial p_k}, \qquad \tilde{D}_t = D_t + \sum_{k\geq 0} \tilde{D}_x^k(p_3 + up_1)\frac{\partial}{\partial p_k}.$$

Then, solving the equation $\ell_{\mathscr{E}}(F) = 0$, that is

$$\tilde{D}_t(F) = \tilde{D}_x^3(F) + u\tilde{D}_x(F) + u_1 F \tag{17}$$

with respect to the function $F = \sum_i F_i p_i$, where $F_i = F_i(x, t, u, \dots, u_k)$, we obtain two independent solutions

$$F^0 = p_1, \qquad F^1 = p_3 + \tfrac{2}{3}up_1 + \tfrac{1}{3}u_1 p_0,$$

to which there correspond two $\mathcal{C}$-differential operators

$$A^0 = D_x, \qquad A^1 = D_x^3 + \tfrac{2}{3}uD_x + \tfrac{1}{3}u_1,$$

the classical Hamiltonian structures for the KdV equation.

6.3. *The Hamiltonianity test*

To demonstrate how the method works, we shall check the Hamiltonianity of the operators A^0 and A^1 in a straightforward way. Obviously, both operators are skew-adjoint.

For A^0, the corresponding bivector $W_0 = W_{A^0}$ is

$$W_0 = F^0 p_0 = p_1 p_0.$$

Since $\delta W_0/\delta u = 0$, we get

$$\mathcal{E}\left(\frac{\delta W_0}{\delta u}\frac{\delta W_0}{\delta p}\right) = 0.$$

For A^1, one has

$$W_1 = F^1 p_0 = (p_3 + \tfrac{2}{3}up_1 + \tfrac{1}{3}u_1 p_0)p_0 = p_3 p_0 + \tfrac{2}{3}up_1 p_0.$$

Consequently

$$\frac{\delta W_1}{\delta u} = \frac{2}{3}p_1 p_0,$$

and

$$\begin{aligned}\frac{\delta W_1}{\delta p} &= \frac{\partial W_1}{\partial p_0} - D_x\frac{\partial W_1}{\partial p_1} - D_x^3\frac{\partial W_1}{\partial p_3} = -\left(p_3 + \frac{2}{3}up_1\right) - D_x\left(\frac{2}{3}up_0\right) - D_x^3(p_0)\\ &= -2p_3 - \frac{4}{3}up_1 - \frac{2}{3}u_1 p_0.\end{aligned}$$

Hence

$$\frac{\delta W_1}{\delta u}\frac{\delta W_1}{\delta p} = \frac{4}{3}p_0 p_1 p_3 = D_x\left(\frac{4}{3}p_0 p_1 p_2\right),$$

that implies $\mathcal{E}((\delta W_1/\delta u)\cdot(\delta W_1/\delta p)) = 0$.

Remark 11. In [6] we describe a class of equations which have the property that (12) automatically implies the Hamiltonianity. In particular, KdV belongs to that class, thus the above verification might be skipped.

6.4. Nonlocal Hamiltonian structure

Let us introduce a new (odd) nonlocal variable determined by the equations

$$r_x = u_1 p_0, \qquad r_t = u_1 p_2 - u_2 p_1 + (uu_1 + u_3) p_0$$

(see Example 5). Then an additional solution of Eq. (17) arises:

$$F^2 = p_5 + \tfrac{4}{3} u p_3 + 2u_1 p_2 + (\tfrac{4}{9} u^2 + \tfrac{4}{3} u_2) p_1 + (\tfrac{4}{9} uu_1 + \tfrac{1}{3} u_3) p_0 - \tfrac{1}{9} u_1 r,$$

to which there corresponds the operator

$$A^2 = D_x^5 + \tfrac{4}{3} u D_x^3 + 2u_1 D_x^2 + (\tfrac{4}{9} u^2 + \tfrac{4}{3} u_2) D_x + (\tfrac{4}{9} uu_1 + \tfrac{1}{3} u_3) - \tfrac{1}{9} u_1 D_x^{-1} \circ u_1.$$

Remark 12. Here and below we use the following correspondence between nonlocal variables and operators (in the case of evolution equations with one-dimensional x). Let p_k^j be the variables in the fibers of the $\ell_{\mathscr{E}}^*$-covering and nonlocal variable r be determined by the relations

$$r_x = \sum_{k,j} a_k^j p_k^j, \qquad r_t = \sum_{k,j} b_k^j p_k^j$$

(cf. Example 5). Then the corresponding operator Δ_r acts on $\varphi = (\varphi^1, \dots, \varphi^m)$ by

$$\Delta_r(\varphi) = D_x^{-1} \left(\sum_{k,j} a_k^j D_x^k(\varphi^j) \right).$$

Simulating the techniques developed for the local case, it is a straightforward check that A^2 is a Hamiltonian structure and all three structures are pair-wise compatible. Moreover, they are related to each other by the classical recursion operator

$$R = D_x^2 + \tfrac{2}{3} u + \tfrac{1}{3} u_1 D_x^{-1},$$

i.e., $A^1 = R \circ A^0$ and $A^2 = R \circ A^1$. In a similar way, one can find a whole infinite series of nonlocal Hamiltonian structures for the Korteweg–de Vries equation.

Remark 13. We stress here the word *simulating* above: at this moment, we do not have a consistent theory of Hamiltonian structures in the nonlocal setting. We hope to develop it elsewhere.

7. The Boussinesq equation

In this section, we shall present, as another illustration of the above developed methods, computation of local and nonlocal Hamiltonian structures for the classical Boussinesq equation. We consider this equation as the system of the form

$$u_t = u_x v + u v_x + \sigma v_{xxx}, \qquad v_t = u_x + v v_x, \tag{18}$$

where $\sigma \in \mathbb{R}$ is a constant.

All computations presented below were done by the software system described in [7, Chapter VIII] and we expose here final results only.

7.1. The $\ell^*_{\mathscr{E}}$-covering

The linearization operator restricted to $\mathscr{E}^\infty$ is

$$\ell_{\mathscr{E}} = \begin{pmatrix} vD_x - D_t & \sigma D_x^3 + u_1 \\ D_x & vD_x + v_1 - D_t \end{pmatrix},$$

while the adjoint one is expressed by

$$\ell^*_{\mathscr{E}} = \begin{pmatrix} -vD_x - v_1 + D_t & -D_x \\ -\sigma D_x^3 + u_1 & -vD_x + D_t \end{pmatrix}.$$

Hence, the $\ell^*_{\mathscr{E}}$-covering with the odd nonlocal variables p_i, q_i is defined by

$$p_t = vp_1 + v_1 p + q_1, \qquad q_t = \sigma p_3 - u_1 p + vq_1.$$

7.2. Local Hamiltonian operators

In a completely similar way as described for the KdV equation in the previous section, we solved the symmetry equation in the $\ell^*_{\mathscr{E}}$-covering of the classical Boussinesq equation. We found three local solutions of the form

$$\begin{aligned}
&F^1 = q_1, \quad G^1 = p_1;\\
&F^2 = 2\sigma p_3 + 2up_1 + u_1 p_0 + vq_1, \quad G^2 = vp_1 + v_1 p_0 + 2q_1;\\
&F^3 = 4\sigma vp_3 + 6\sigma v_1 p_2 + 2(3\sigma v_2 + 2uv)p_1 + 2(\sigma v_3 + uv_1 + u_1 v)p_0\\
&\qquad + 4\sigma q_3 + (4u + v^2)q_1 + 2u_1 q_0,\\
&G^3 = 4\sigma p_3 + (4u + v^2)p_1 + 2(u_1 + vv_1)p_0 + 4vq_1 + 2v_1 q_0.
\end{aligned}$$

In classical operator notation, they are represented as

$$A^1 = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix}, \qquad A^2 = \begin{pmatrix} 2\sigma D_x^3 + 2uD_x + u_1 & vD_x \\ vD_x + v_1 & 2D_x \end{pmatrix},$$

while the third operator has the entries

$$\begin{aligned}
&A^3_{11} = 4\sigma vD_x^3 + 6\sigma v_1 D_x^2 + 2(3\sigma v_2 + 2uv)D_x + 2(\sigma v_3 + uv_1 + u_1 v),\\
&A^3_{12} = 4\sigma D_x^3 + (4u + v^2)D_x + 2u_1, \quad A^3_{21} = 4\sigma D_x^3 + (4u + v^2)D_x + 2(u_1 + vv_1),\\
&A^3_{22} = 4vD_x + 2v_1.
\end{aligned}$$

7.3. Hamiltonianity and compatibility

To test Hamiltonianity and compatibility conditions for the operators A^1, A^2, A^3, we construct the bivectors

$$
\begin{aligned}
W_{A^1} &= q_1 p_0 + p_1 q_0,\\
W_{A^2} &= (2\sigma p_3 + 2u p_1 + u_1 p_0 + v q_1) p_0 + (v p_1 + v_1 p_0 + 2 q_1) q_0,\\
W_{A^3} &= (4\sigma v p_3 + 6\sigma v_1 p_2 + 2(3\sigma v_2 + 2uv) p_1 + 2(\sigma v_3 + u v_1 + u_1 v) p_0\\
&\quad + 4\sigma q_3 + (4u + v^2) q_1 + 2u_1 q_0) p_0 + (4\sigma p_3 + (4u + v^2) p_1\\
&\quad + 2(u_1 + v v_1) p_0 + 4 v q_1 + 2 v_1 q_0) q_0,
\end{aligned}
$$

and straightforwardly check that

$$
[\![W_{A^i}, W_{A^j}]\!] = 0, \quad 1 \le i \le j \le 3,
$$

i.e., the operators A^1, A^2, A^3 meet both Hamiltonianity and compatibility conditions.

7.4. Nonlocal Hamiltonian operators

In order to describe nonlocal results we introduce three new nonlocal variables r_1, r_2, r_3 over the $\ell^*_{\mathscr{E}}$-covering by the following definitions:

$$
\begin{aligned}
r_{1,x} &= p_0 u_1 + q_0 v_1, \quad r_{1,t} = p_2 \sigma v_1 - p_1 \sigma v_2 + p_0(\sigma v_3 + u v_1 + u_1 v) + q_0(u_1 + v v_1);\\
r_{2,x} &= p_0(\sigma v_3 + u v_1 + u_1 v) + q_0(u_1 + v v_1),\\
r_{2,t} &= p_2 \sigma(u_1 + v v_1) - p_1 \sigma(u_2 + v v_2 + v_1^2) + p_0(\sigma u_3 + 2\sigma v v_3 + 3\sigma v_1 v_2 + u u_1\\
&\quad + 2u v v_1 + u_1 v^2) + q_0(\sigma v_3 + u v_1 + 2 u_1 v + v^2 v_1);\\
r_{3,x} &= p_0(4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 6 u u_1 + 6 u v v_1 + 3 u_1 v^2)\\
&\quad + q_0(4\sigma v_3 + 6 u v_1 + 6 u_1 v + 3 v^2 v_1),\\
r_{3,t} &= p_2 \sigma(4\sigma v_3 + 6 u v_1 + 6 u_1 v + 3 v^2 v_1) + p_1 \sigma(-4\sigma v_4 - 6 u v_2 - 12 u_1 v_1 - 6 u_2 v\\
&\quad - 3 v^2 v_2 - 6 v v_1^2) + p_0(4\sigma^2 v_5 + 10\sigma u v_3 + 18\sigma u_1 v_2 + 18\sigma u_2 v_1 + 10\sigma u_3 v\\
&\quad + 9\sigma v^2 v_3 + 30\sigma v v_1 v_2 + 6\sigma v_1^3 + 6u^2 v_1 + 12 u u_1 v + 9 u v^2 v_1 + 3 u_1 v^3)\\
&\quad + q_0(4\sigma u_3 + 10\sigma v v_3 + 12\sigma v_1 v_2 + 6 u u_1 + 12 u v v_1 + 9 u_1 v^2 + 3 v^3 v_1).
\end{aligned}
$$

Using these nonlocal variables, we derived the following three nonlocal Hamiltonian structures given by

$$
\begin{aligned}
F^4 &= 8\sigma^2 p_5 + 2\sigma(8u + 3v^2) p_3 + 6\sigma(4u_1 + 3 v v_1) p_2 + 2(8\sigma u_2 + 9\sigma v v_2 + 6\sigma v_1^2\\
&\quad + 4u^2 + 3 u v^2) p_1 + (4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 8 u u_1 + 6 u v v_1 + 3 u_1 v^2) p_0\\
&\quad + 12\sigma v q_3 + 20\sigma v_1 q_2 + (16\sigma v_2 + 12 u v + v^3) q_1\\
&\quad + 2(2\sigma v_3 + 2 u v_1 + 3 u_1 v) q_0 - 2 u_1 r_1,\\
G^4 &= 12\sigma v p_3 + 16\sigma v_1 p_2 + (12\sigma v_2 + 12 u v + v^3) p_1 + (4\sigma v_3 + 8 u v_1 + 6 u_1 v\\
&\quad + 3 v^2 v_1) p_0 + 8\sigma q_3 + 2(4u + 3v^2) q_1 + 2(2u_1 + 3 v v_1) q_0 - 2 v_1 r_1;
\end{aligned}
$$

$$
\begin{aligned}
F^5 = {} & 32\sigma^2 v p_5 + 80\sigma^2 v_1 p_4 + 8\sigma(14\sigma v_2 + 8uv + v^3) p_3 + 4\sigma(22\sigma v_3 + 24uv_1 \\
& + 24u_1 v + 9v^2 v_1) p_2 + 4(10\sigma^2 v_4 + 20\sigma u v_2 + 26\sigma u_1 v_1 + 16\sigma u_2 v + 9\sigma v^2 v_2 \\
& + 12\sigma v v_1^2 + 8u^2 v + 2uv^3) p_1 + 4(2\sigma^2 v_5 + 6\sigma u v_3 + 11\sigma u_1 v_2 + 9\sigma u_2 v_1 \\
& + 4\sigma u_3 v + 3\sigma v^2 v_3 + 12\sigma v v_1 v_2 + 3\sigma v_1^3 + 4u^2 v_1 + 8uu_1 v + 3uv^2 v_1 + u_1 v^3) p_0 \\
& + 16\sigma^2 q_5 + 8\sigma(4u + 3v^2) q_3 + 16\sigma(3u_1 + 5vv_1) q_2 + (32\sigma u_2 + 64\sigma v v_2 \\
& + 44\sigma v_1^2 + 16u^2 + 24uv^2 + v^4) q_1 + 4(2\sigma u_3 + 4\sigma v v_3 + 6\sigma v_1 v_2 + 4uu_1 \\
& + 4uvv_1 + 3u_1 v^2) q_0 - 4u_1 r_2 - 4(\sigma v_3 + uv_1 + u_1 v) r_1, \\
G^5 = {} & 16\sigma^2 p_5 + 8\sigma(4u + 3v^2) p_3 + 16\sigma(3u_1 + 4vv_1) p_2 + (32\sigma u_2 + 48\sigma v v_2 \\
& + 28\sigma v_1^2 + 16u^2 + 24uv^2 + v^4) p_1 + 4(2\sigma u_3 + 4\sigma v v_3 + 8\sigma v_1 v_2 + 4uu_1 \\
& + 8uvv_1 + 3u_1 v^2 + v^3 v_1) p_0 + 32\sigma v q_3 + 48\sigma v_1 q_2 + 8(4\sigma v_2 + 4uv + v^3) q_1 \\
& + 4(2\sigma v_3 + 4uv_1 + 4u_1 v + 3v^2 v_1) q_0 - 4v_1 r_2 - 4(u_1 + vv_1) r_1; \\
F^6 = {} & -32\sigma^3 p_7 - 16\sigma^2(6u + 5v^2) p_5 - 80\sigma^2(3u_1 + 5vv_1) p_4 \\
& - 2\sigma(160\sigma u_2 + 280\sigma v v_2 + 204\sigma v_1^2 + 48u^2 + 80uv^2 + 5v^4) p_3 \\
& - 4\sigma(60\sigma u_3 + 110\sigma v v_3 + 216\sigma v_1 v_2 + 72uu_1 + 120uvv_1 + 60u_1 v^2 \\
& + 15v^3 v_1) p_2 - 2(48\sigma^2 u_4 + 100\sigma^2 v v_4 + 244\sigma^2 v_1 v_3 + 168\sigma^2 v_2^2 + 96\sigma u u_2 \\
& + 200\sigma u v v_2 + 136\sigma u v_1^2 + 68\sigma u_1^2 + 260\sigma u_1 v v_1 + 80\sigma u_2 v^2 + 30\sigma v^3 v_2 \\
& + 60\sigma v^2 v_1^2 + 16u^3 + 40u^2 v^2 + 5uv^4) p_1 - (16\sigma^2 u_5 + 40\sigma^2 v v_5 + 120\sigma^2 v_1 v_4 \\
& + 208\sigma^2 v_2 v_3 + 48\sigma u u_3 + 120\sigma u v v_3 + 232\sigma u v_1 v_2 + 88\sigma u_1 u_2 + 220\sigma u_1 v v_2 \\
& + 156\sigma u_1 v_1^2 + 180\sigma u_2 v v_1 + 40\sigma u_3 v^2 + 20\sigma v^3 v_3 + 120\sigma v^2 v_1 v_2 + 60\sigma v v_1^3 \\
& + 48u^2 u_1 + 80u^2 v v_1 + 80uu_1 v^2 + 20uv^3 v_1 + 5u_1 v^4) p_0 - 80\sigma^2 v q_5 \\
& - 224\sigma^2 v_1 q_4 - 40\sigma(8\sigma v_2 + 4uv + v^3) q_3 - 8\sigma(30\sigma v_3 + 32uv_1 + 30u_1 v \\
& + 25v^2 v_1) q_2 - (96\sigma^2 v_4 + 192\sigma u v_2 + 256\sigma u_1 v_1 + 160\sigma u_2 v + 160\sigma v^2 v_2 \\
& + 220\sigma v v_1^2 + 80u^2 v + 40uv^3 + v^5) q_1 - 4(4\sigma^2 v_5 + 12\sigma u v_3 + 22\sigma u_1 v_2 \\
& + 18\sigma u_2 v_1 + 10\sigma u_3 v + 10\sigma v^2 v_3 + 30\sigma v v_1 v_2 + 6\sigma v_1^3 + 8u^2 v_1 + 20uu_1 v \\
& + 10uv^2 v_1 + 5u_1 v^3) q_0 + 2u_1 r_3 + 8(\sigma v_3 + uv_1 + u_1 v) r_2 + 2(4\sigma u_3 + 6\sigma v v_3 \\
& + 12\sigma v_1 v_2 + 6uu_1 + 6uvv_1 + 3u_1 v^2) r_1, \\
G^6 = {} & -80\sigma^2 v p_5 - 176\sigma^2 v_1 p_4 - 8\sigma(28\sigma v_2 + 20uv - 5v^3) p_3 - 16\sigma(11\sigma v_3 + 14uv_1 \\
& + 15u_1 v + 10v^2 v_1) p_2 - (80\sigma^2 v_4 + 160\sigma u v_2 + 224\sigma u_1 v_1 + 160\sigma u_2 v \\
& + 120\sigma v^2 v_2 + 140\sigma v v_1^2 + 80u^2 v + 40uv^3 + v^5) p_1 - (16\sigma^2 v_5 + 48\sigma u v_3 \\
& + 88\sigma u_1 v_2 + 88\sigma u_2 v_1 + 40\sigma u_3 v + 40\sigma v^2 v_3 + 160\sigma v v_1 v_2 + 36\sigma v_1^3 + 48u^2 v_1 \\
& + 80uu_1 v + 80uv^2 v_1 + 20u_1 v^3 + 5v^4 v_1) p_0 - 32\sigma^2 q_5 - 16\sigma(4u + 5v^2) q_3 \\
& - 48\sigma(2u_1 + 5vv_1) q_2 - 2(32\sigma u_2 + 80\sigma v v_2 + 52\sigma v_1^2 + 16u^2 + 40uv^2 + 5v^4) q_1 \\
& - 4(4\sigma u_3 + 10\sigma v v_3 + 16\sigma v_1 v_2 + 8uu_1 + 20uvv_1 + 10u_1 v^2 + 5v^3 v_1) q_0 \\
& + 2v_1 r_3 + 8(u_1 + vv_1) r_2 + 2(4\sigma v_3 + 6uv_1 + 6u_1 v + 3v^2 v_1) r_1.
\end{aligned}
$$

In the conventional notation the operators A^4, A^5, and A^6 have the following entries:

$$\begin{aligned}
A^4_{11} &= 8\sigma^2 D_x^5 + 2\sigma(8u + 3v^2)D_x^3 + 6\sigma(4u_1 + 3vv_1)D_x^2 + 2(8\sigma u_2 + 9\sigma vv_2 + 6\sigma v_1^2 \\
&\quad + 4u^2 + 3uv^2)D_x^1 + (4\sigma u_3 + 6\sigma vv_3 + 12\sigma v_1 v_2 + 8uu_1 + 6uvv_1 + 3u_1 v^2) \\
&\quad - 2u_1 D_x^{-1} \circ u_1, \\
A^4_{12} &= 12\sigma v D_x^3 + 20\sigma v_1 D_x^2 + (16\sigma v_2 + 12uv + v^3)D_x + 2(2\sigma v_3 + 2uv_1 + 3u_1 v) \\
&\quad -2u_1 D_x^{-1} \circ v_1, \\
A^4_{21} &= 12\sigma v D_x^3 + 16\sigma v_1 D_x^2 + (12\sigma v_2 + 12uv + v^3)D_x \\
&\quad + (4\sigma v_3 + 8uv_1 + 6u_1 v + 3v^2 v_1) - 2v_1 D_x^{-1} \circ u_1, \\
A^4_{22} &= 8\sigma D_x^3 + 2(4u + 3v^2)D_x + 2(2u_1 + 3vv_1) - 2v_1 D_x^{-1} \circ v_1.
\end{aligned}$$

The matrix elements of A^5 are given as

$$\begin{aligned}
A^5_{11} &= 32\sigma^2 v D_x^5 + 80\sigma^2 v_1 D_x^4 + 8\sigma(14\sigma v_2 + 8uv + v^3)D_x^3 + 4\sigma(22\sigma v_3 + 24uv_1 \\
&\quad + 24u_1 v + 9v^2 v_1)D_x^2 + 4(10\sigma^2 v_4 + 20\sigma uv_2 + 26\sigma u_1 v_1 + 16\sigma u_2 v + 9\sigma v^2 v_2 \\
&\quad + 12\sigma vv_1^2 + 8u^2 v + 2uv^3)D_x + 4(2\sigma^2 v_5 + 6\sigma uv_3 + 11\sigma u_1 v_2 + 9\sigma u_2 v_1 \\
&\quad + 4\sigma u_3 v + 3\sigma v^2 v_3 + 12\sigma vv_1 v_2 + 3\sigma v_1^3 + 4u^2 v_1 + 8uu_1 v + 3uv^2 v_1 + u_1 v^3) \\
&\quad - 4(\sigma v_3 + uv_1 + u_1 v)D_x^{-1} \circ u_1 - 4u_1 D_x^{-1} \circ (\sigma v_3 + uv_1 + u_1 v), \\
A^5_{12} &= 16\sigma^2 D_x^5 + 8\sigma(4u + 3v^2)D_x^3 + 16\sigma(3u_1 + 5vv_1)D_x^2 + (32\sigma u_2 + 64\sigma vv_2 \\
&\quad + 44\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)D_x + 4(2\sigma u_3 + 4\sigma vv_3 + 6\sigma v_1 v_2 + 4uu_1 \\
&\quad + 4uvv_1 + 3u_1 v^2) - 4(\sigma v_3 + uv_1 + u_1 v)D_x^{-1} \circ v_1 - 4u_1 D_x^{-1} \circ (u_1 + vv_1), \\
A^5_{21} &= +16\sigma^2 D_x^5 + 8\sigma(4u + 3v^2)D_x^3 + 16\sigma(3u_1 + 4vv_1)D_x^2 + (32\sigma u_2 + 48\sigma vv_2 \\
&\quad + 28\sigma v_1^2 + 16u^2 + 24uv^2 + v^4)D_x + 4(2\sigma u_3 + 4\sigma vv_3 + 8\sigma v_1 v_2 + 4uu_1 \\
&\quad + 8uvv_1 + 3u_1 v^2 + v^3 v_1) - 4(u_1 + vv_1)D_x^{-1} \circ u_1 \\
&\quad - 4v_1 D_x^{-1} \circ (\sigma v_3 + uv_1 + u_1 v), \\
A^5_{22} &= 32\sigma v D_x^3 + 48\sigma v_1 D_x^2 + 8(4\sigma v_2 + 4uv + v^3)D_x + 4(2\sigma v_3 + 4uv_1 \\
&\quad + 4u_1 v + 3v^2 v_1) - 4(u_1 + vv_1)D_x^{-1} \circ v_1 - 4v_1 D_x^{-1} \circ (u_1 + vv_1).
\end{aligned}$$

The matrix elements of A^6 are given as

$$\begin{aligned}
A^6_{11} &= -32\sigma^3 D_x^7 + 16\sigma^2(-6u - 5v^2)D_x^5 + 80\sigma^2(-3u_1 - 5vv_1)D_x^4 \\
&\quad + 2\sigma(-160\sigma u_2 - 280\sigma vv_2 - 204\sigma v_1^2 - 48u^2 - 80uv^2 - 5v^4)D_x^3 \\
&\quad + 4\sigma(-60\sigma u_3 - 110\sigma vv_3 - 216\sigma v_1 v_2 - 72uu_1 - 120uvv_1 \\
&\quad - 60u_1 v^2 - 15v^3 v_1)D_x^2 + 2(-48\sigma^2 u_4 - 100\sigma^2 vv_4 - 244\sigma^2 v_1 v_3 - 168\sigma^2 v_2^2 \\
&\quad - 96\sigma uu_2 - 200\sigma uvv_2 - 136\sigma uv_1^2 - 68\sigma u_1^2 - 260\sigma u_1 vv_1 - 80\sigma u_2 v^2 \\
&\quad - 30\sigma v^3 v_2 - 60\sigma v^2 v_1^2 - 16u^3 - 40u^2 v^2 - 5uv^4)D_x + (-16\sigma^2 u_5 - 40\sigma^2 vv_5 \\
&\quad - 120\sigma^2 v_1 v_4 - 208\sigma^2 v_2 v_3 - 48\sigma uu_3 - 120\sigma uvv_3 - 232\sigma uv_1 v_2 - 88\sigma u_1 u_2 \\
&\quad - 220\sigma u_1 vv_2 - 156\sigma u_1 v_1^2 - 180\sigma u_2 vv_1 - 40\sigma u_3 v^2 - 20\sigma v^3 v_3 - 120\sigma v^2 v_1 v_2
\end{aligned}$$

$$
\begin{aligned}
&- 60\sigma v v_1^3 - 48u^2u_1 - 80u^2vv_1 - 80uu_1v^2 - 20uv^3v_1 - 5u_1v^4) \\
&+ 2(4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 6uu_1 + 6uvv_1 + 3u_1v^2)D_x^{-1} \circ u_1 \\
&+ 8(\sigma v_3 + uv_1 + u_1 v)D_x^{-1} \circ (\sigma v_3 + uv_1 + u_1 v) + 2u_1 D_x^{-1} \circ (4\sigma u_3 + 6\sigma v v_3 \\
&+ 12\sigma v_1 v_2 + 6uu_1 + 6uvv_1 + 3u_1 v^2),
\end{aligned}
$$

$$
\begin{aligned}
A_{12}^6 = {} & -80\sigma^2 v D_x^5 - 224\sigma^2 v_1 D_x^4 + 40\sigma(-8\sigma v_2 - 4uv - v^3)D_x^3 + 8\sigma(-30\sigma v_3 \\
&- 32uv_1 - 30u_1 v - 25v^2 v_1)D_x^2 + (-96\sigma^2 v_4 - 192\sigma u v_2 - 256\sigma u_1 v_1 \\
&- 160\sigma u_2 v - 160\sigma v^2 v_2 - 220\sigma v v_1^2 - 80u^2 v - 40uv^3 - v^5)D_x + 4(-4\sigma^2 v_5 \\
&- 12\sigma u v_3 - 22\sigma u_1 v_2 - 18\sigma u_2 v_1 - 10\sigma u_3 v - 10\sigma v^2 v_3 - 30\sigma v v_1 v_2 - 6\sigma v_1^3 \\
&- 8u^2 v_1 - 20uu_1 v - 10uv^2 v_1 - 5u_1 v^3) + 2(4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 \\
&+ 6uu_1 + 6uvv_1 + 3u_1 v^2)D_x^{-1} \circ v_1 + 8(\sigma v_3 + uv_1 + u_1 v)D_x^{-1} \circ (u_1 + vv_1) \\
&+ 2u_1 D_x^{-1} \circ (4\sigma v_3 + 6uv_1 + 6u_1 v + 3v^2 v_1),
\end{aligned}
$$

$$
\begin{aligned}
A_{21}^6 = {} & -80\sigma^2 v D_x^5 - 176\sigma^2 v_1 D_x^4 + 8\sigma(-28\sigma v_2 - 20uv - 5v^3)D_x^3 + 16\sigma(-11\sigma v_3 \\
&- 14uv_1 - 15u_1 v - 10v^2 v_1)D_x^2 + (-80\sigma^2 v_4 - 160\sigma u v_2 - 224\sigma u_1 v_1 \\
&- 160\sigma u_2 v - 120\sigma v^2 v_2 - 140\sigma v v_1^2 - 80u^2 v - 40uv^3 - v^5)D_x + (-16\sigma^2 v_5 \\
&- 48\sigma u v_3 - 88\sigma u_1 v_2 - 88\sigma u_2 v_1 - 40\sigma u_3 v - 40\sigma v^2 v_3 - 160\sigma v v_1 v_2 \\
&- 36\sigma v_1^3 - 48u^2 v_1 - 80uu_1 v - 80uv^2 v_1 - 20u_1 v^3 - 5v^4 v_1) + 2(4\sigma v_3 + 6uv_1 \\
&+ 6u_1 v + 3v^2 v_1)D_x^{-1} \circ u_1 + 8(u_1 + vv_1)D_x^{-1} \circ (\sigma v_3 + uv_1 + u_1 v) \\
&+ 2v_1 D_x^{-1} \circ (4\sigma u_3 + 6\sigma v v_3 + 12\sigma v_1 v_2 + 6uu_1 + 6uvv_1 + 3u_1 v^2),
\end{aligned}
$$

$$
\begin{aligned}
A_{22}^6 = {} & -32\sigma^2 D_x^5 + 16\sigma(-4u - 5v^2)D_x^3 + 48\sigma(-2u_1 - 5vv_1)D_x^2 + 2(-32\sigma u_2 \\
&- 80\sigma v v_2 - 52\sigma v_1^2 - 16u^2 - 40uv^2 - 5v^4)D_x + 4(-4\sigma u_3 - 10\sigma v v_3 \\
&- 16\sigma v_1 v_2 - 8uu_1 - 20uvv_1 - 10u_1 v^2 - 5v^3 v_1) + 2(4\sigma v_3 + 6uv_1 + 6u_1 v \\
&+ 3v^2 v_1)D_x^{-1} \circ v_1 + 8(u_1 + vv_1)D_x^{-1} \circ (u_1 + vv_1) \\
&+ 2v_1 D_x^{-1} \circ (4\sigma v_3 + 6uv_1 + 6u_1 v + 3v^2 v_1).
\end{aligned}
$$

Similar to the previous cases, we checked the conditions for Hamiltonianity and compatibility of all six Hamiltonian structures. It is also easy to check that all six structures are related by the recursion operator constructed for symmetries of the Boussinesq equation, see, e.g. [7].

8. The coupled KdV–mKdV system

We shall now describe a Hamiltonian structure for the coupled KdV–mKdV system of the form

$$
\begin{aligned}
u_t &= -u_{xxx} + 6uu_x - 3vv_{xxx} - 3v_x v_{xx} + 3u_x v^2 + 6uvv_x, \\
v_t &= -v_{xxx} + 3v^2 v_x + 3uv_x + 3u_x v.
\end{aligned}
\tag{19}
$$

This system arises as the so-called *bosonic limit* of the $N = 2$, $a = 1$ supersymmetric extension of the KdV equation [4]; integrability properties of this system (existence of a recursion operator) were studied in [5]. In [3], by means of the prolongations structure techniques, a Lax pair for (19) was constructed.

Denote the evolution equation corresponding to (19) by $\mathcal{E}^\infty$ and choose for coordinates in $\mathcal{E}^\infty$ the functions

$$x, t, u = u_0, v = v_0, \dots, u_k, v_k, \dots$$

Then the total derivative operators restricted to $\mathcal{E}^\infty$ are written in the form

$$D_x = \frac{\partial}{\partial x} + \sum_{k\geq 0}\left(u_{k+1}\frac{\partial}{\partial u_k} + v_{k+1}\frac{\partial}{\partial v_k}\right),$$
$$D_t = \frac{\partial}{\partial t} + \sum_{k\geq 0}\left(D_x^k(f)\frac{\partial}{\partial u_k} + D_x^k(g)\frac{\partial}{\partial v_k}\right), \tag{20}$$

where

$$f = -u_3 + 6uu_1 - 3vv_3 - 3v_1v_2 + 3u_1v^2 + 6uvv_1,$$
$$g = -v_3 + 3v^2v_1 + 3uv_1 + 3u_1v$$

are the functions at the right-hand side of (19).

8.1. *The $\ell^*_{\mathcal{E}}$-covering*

The linearization operator restricted to $\mathcal{E}^\infty$ is

$$\ell_{\mathcal{E}} = \begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix}, \tag{21}$$

where

$$\ell_{11} = D_t + D_x^3 - (6u + 3v^2)D_x - 6(u_1 + vv_1),$$
$$\ell_{12} = 3vD_x^3 + 3v_1D_x^2 - (6uv - 3v_2)D_x - 6u_1v - 6uv_1 + 3v_3,$$
$$\ell_{21} = -3vD_x - 3v_1, \qquad \ell_{22} = D_t + D_x^3 - 3(u + v^2)D_x - (6vv_1 + 3u).$$

Consequently, the adjoint operator is

$$\ell^*_{\mathcal{E}} = \begin{pmatrix} \ell^*_{11} & \ell^*_{21} \\ \ell^*_{12} & \ell^*_{22} \end{pmatrix}, \tag{22}$$

where

$$\ell^*_{11} = -D_t - D_x^3 + (6u + 3v^2)D_x, \qquad \ell^*_{21} = 3vD_x,$$
$$\ell^*_{12} = -3vD_x^3 - 6v_1D_x^2 + 6(uv - v_2)D_x, \qquad \ell^*_{22} = -D_t - D_x^3 + 3(u + v^2)D_x.$$

Following the general theory of Section 4, we now construct the $\ell^*_{\mathscr{E}}$-covering for the equation $\mathscr{E}^\infty$ by introducing new odd variables $p = p_0, q = q_0, \dots, p_k, q_k, \dots, p_k = D_x^k(p), q_k = D_x^k(q)$, that obey the equations

$$p_t = -p_3 + (6u + 3v^2)p_1 + 3vq_1, \tag{23}$$

$$q_t = -3vp_3 - 6v_1p_2 + 6(uv - v_2)p_1 - q_3 + 3(u + v^2)q_1. \tag{24}$$

8.2. *Solving the defining equations*

We now introduce a vector function of the form

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} \sum_i (F_i^u p_i + F_i^v q_i) \\ \sum_i (G_i^u p_i + G_i^v q_i) \end{pmatrix},$$

where F_i^u, F_i^v, G_i^u, G_i^v are functions on $\mathscr{E}^\infty$, and solve the equation

$$\begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = 0. \tag{25}$$

The operators ℓ_{ij} here are lifted to the $\ell^*_{\mathscr{E}}$-covering, which means that the total derivatives are now of the form

$$\tilde{D}_x = \frac{\partial}{\partial x} + \sum_{k\geq 0} \left(u_{k+1}\frac{\partial}{\partial u_k} + v_{k+1}\frac{\partial}{\partial v_k} + p_{k+1}\frac{\partial}{\partial p_k} + q_{k+1}\frac{\partial}{\partial q_k} \right),$$
$$\tilde{D}_t = \frac{\partial}{\partial t} + \sum_{k\geq 0} \left(D_x^k(f)\frac{\partial}{\partial u_k} + D_x^k(g)\frac{\partial}{\partial v_k} + D_x^k(f')\frac{\partial}{\partial p_k} + D_x^k(g')\frac{\partial}{\partial q_k} \right), \tag{26}$$

where f' and g' are the right-hand sides of (23) and (24), respectively.

The following solution was obtained:

$$F = -p_3 + 4up_1 + 2u_1p_0 + 2vq_1, \qquad G = 2vp_1 + 2v_1p_0 + q_1,$$

to which there corresponds the operator

$$A = \begin{pmatrix} -D_x^3 + 4uD_x + 2u_1 & 2vD_x \\ 2vD_x + 2v_1 & D_x \end{pmatrix}. \tag{27}$$

8.3. *The Hamiltonianity test*

We shall check now that the operator A presented by (27) is Hamiltonian. The first property is obvious: evidently, $A^* = -A$, i.e., A is a skew-adjoint operator.

To check the second property, we construct the bivector

$$\begin{aligned} W_A &= Fp_0 + Gq_0 = (-p_3 + 4up_1 + 2u_1p_0 + 2vq_1)p_0 + (2vp_1 + 2v_1p_0 + q_1)q_0 \\ &= p_0p_3 - 4up_0p_1 - 2vp_0q_1 + 2vp_1q_0 + 2v_1p_0q_0 - q_0q_1, \end{aligned}$$

and verify condition (9), i.e.

$$\mathcal{E}\left(\frac{\delta W_A}{\delta u}\frac{\delta W_A}{\delta p}+\frac{\delta W_A}{\delta v}\frac{\delta W_A}{\delta q}\right)=0. \tag{28}$$

But

$$\frac{\delta W_A}{\delta u}=-4p_0p_1, \qquad \frac{\delta W_A}{\delta v}=-4p_0q_1,$$
$$\frac{\delta W_A}{\delta p}=2(p_3-4up_1-2u_1p_0-2vq_1), \qquad \frac{\delta W_A}{\delta q}=2(-2vp_1-2v_1p_0-q_1),$$

and consequently

$$\frac{\delta W_A}{\delta u}\frac{\delta W_A}{\delta p}+\frac{\delta W_A}{\delta v}\frac{\delta W_A}{\delta q}=-8p_0p_1p_3=D_x(-8p_0p_1p_2),$$

i.e., (28) holds.

8.4. *Existence of a Hamiltonian*

Let us show that the KdV–mKdV system (19) possesses a Hamiltonian, i.e., its right-hand side may be represented in the form

$$\begin{pmatrix} f \\ g \end{pmatrix}=A\mathcal{E}(X), \tag{29}$$

where A is the Hamiltonian operator described above and X is the dx-component of a conservation law $\eta=X\,\mathrm{d}x+T\,\mathrm{d}t$ (the energy).

We computed directly several conservation laws of lower order and obtained the following results (for the sake of briefness, we omit the corresponding dt-components):

$$\eta_1: X=v, \qquad \eta_2: X=u, \qquad \eta_4: X=\tfrac{1}{2}(u^2+uv^2-vv_2),$$
$$\eta_6: X=12u^3+24u^2v^2-6uu_2+6uv^4-30uvv_2-3u_2v^2-8v^3v_2+6vv_4.$$

Generating functions corresponding to these conservation laws, that is, vector functions of the form

$$\begin{pmatrix} \varphi \\ \psi \end{pmatrix}=\mathcal{E}(X)=\begin{pmatrix} \dfrac{\delta X}{\delta u} \\[2mm] \dfrac{\delta X}{\delta v} \end{pmatrix}$$

are

$$\varphi_1=0, \quad \psi_1=1; \qquad \varphi_2=1, \quad \psi_2=0; \qquad \varphi_4=u+\tfrac{1}{2}v^2, \quad \psi_4=uv-v_2;$$
$$\varphi_6=6(6u^2+8uv^2-2u_2+v^4-6vv_2-v_1^2),$$
$$\psi_6=12(4u^2v+2uv^3-5uv_2-5u_1v_1-3u_2v-4v^2v_2-4vv_1^2+v_4).$$

Applying A to $\mathcal{E}(X)$, where X corresponds to η_4, we see that (29) holds.

Theorem 2. *The coupled KdV–mKdV system (19) is Hamiltonian with respect to the Hamiltonian operator (27) and possesses the Hamiltonian* $X = (u^2 + uv^2 - vv_2)/2$. *The corresponding energy is given by the form*

$$\eta = \tfrac{1}{2}(u^2 + uv^2 - vv_2)\,\mathrm{d}x + \tfrac{1}{2}(4u^3 + 9u^2v^2 - 2uu_2 + 3uv^4 - 11uvv_2 + uv_1^2 + u_1^2 \\ -u_1vv_1 - 4u_2v^2 - 6v^3v_2 - 3v^2v_1^2 + vv_4 - v_1v_3 + v_2^2)\,\mathrm{d}t.$$

This structure is unique in the class of Hamiltonian structures independent of x *and* t *and polynomial in* u_k, v_k.

Proof of uniqueness. Let us first note that Eq. (19) admits a scaling symmetry that allows to assign *gradings* to all variables x, t, u_k, and v_k:

$$|x| = -1, \qquad |t| = -3, \qquad |u_k| = k + 2, \qquad |v_k| = k + 1,$$

the grading of a monomial is the sum of gradings of the factors entering this monomial. In particular, $|f| = 5, |g| = 4$. All constructions are in agreement with these gradings and we may restrict computations to homogeneous components. Since the grading of the expression $B\mathcal{E}(X)$ is $|B| + |\mathcal{E}(X)|$, we conclude that the grading of the generating function $\mathcal{E}(X)$ is less than that of the right-hand side of (19). This fact restricts the choice of possible Hamiltonians just to several ones and by a direct computation we find that the only possible solution is given in Theorem 2. □

8.5. Discussion: nonlocalities

In spite of the previous result, we have constructed another Hamiltonian operator for the system under consideration. This operator exists in an appropriate nonlocal setting. First, we introduce a new nonlocal variable w defined by

$$w_x = v, \qquad w_t = 3uv + v^3 - v_2,$$

and corresponding to the conservation law η_1 (see Example 5).

In this nonlocal setting, it is possible to extend the $\ell^*_{\mathcal{E}}$-covering by adding odd nonlocal variables r_1, r_2, r_3 defined by the relations

$$r_{1,x} = q_0v_1 + p_0u_1,$$

$$r_{1,t} = -q_2v_1 + p_2(-u_1 - 3vv_1) + q_1v_2 + p_1(u_2 + 3vv_2 - 3v_1^2) + q_0(3uv_1 + 3u_1v \\ + 3v^2v_1 - v_3) + p_0(6uu_1 + 6uvv_1 + 3u_1v^2 - u_3 - 3vv_3 - 3v_1v_2);$$

$$r_{2,x} = \tfrac{1}{2}q_0\cos(2w)v_1 - \tfrac{1}{2}q_0\sin(2w)u - p_0\cos(w)(\tfrac{1}{2}u_1 + vv_1) \\ + p_0\sin(2w)(uv - \tfrac{1}{2}v_2),$$

$$r_{2,t} = -\tfrac{1}{2}q_2\cos(2w)v_1 + \tfrac{1}{2}q_2\sin(2w)u + \tfrac{1}{2}p_2\cos(2w)(u_1 - vv_1) \\ + \tfrac{1}{2}p_2\sin(2w)(uv + v_2) - q_1\cos(2w)(uv + \tfrac{1}{2}v_2) - q_1\sin(2w)(\tfrac{1}{2}u_1 + vv_1)$$

$$
\begin{aligned}
&- p_1\cos(2w)(uv^2+\tfrac12u_2+\tfrac12vv_2+\tfrac52v_1^2)+p_1\sin(2w)(\tfrac52uv_1+\tfrac12u_1v\\
&- v^2v_1-\tfrac12v_3)+\tfrac12q_0\cos(2w)(5uv_1+u_1v+v^2v_1-v_3)+q_0\sin(2w)(-\tfrac32u^2\\
&- \tfrac12uv^2+\tfrac12u2+\tfrac12vv_2+v_1^2)+p_0\cos(2w)(-3uu_1-6uvv_1-\tfrac32u_1v^2+\tfrac12u_3\\
&- v^3v_1+\tfrac32vv_3+\tfrac52v_1v_2)+p_0\sin(2w)(3u^2v+uv^3-\tfrac52uv_2-3u_1v_1\\
&- \tfrac32u_2v-\tfrac32v^2v_2-3vv_1^2+\tfrac12v_4);\\
r_{3,x} ={}& -\tfrac12q_0\cos(2w)u-\tfrac12q_0\sin(2w)v_1+p_0\cos(2w)(uv-\tfrac12v_2)\\
&+ p_0\sin(2w)(\tfrac12u_1+vv_1),\\
r_{3,t} ={}& \tfrac12q_2\cos(2w)u+\tfrac12q_2\sin(2w)v_1+\tfrac12p_2\cos(2w)(uv+v_2)\\
&+ \tfrac12p_2\sin(2w)(-u_1+vv_1)-q_1\cos(2w)(\tfrac12u_1+vv_1)\\
&+ q_1\sin(2w)(uv-\tfrac12v_2)+p_1\cos(2w)(\tfrac52uv_1+\tfrac12u_1v-v^2v_1-\tfrac12v_3)\\
&+ p_1\sin(2w)(uv^2+\tfrac12u_2+\tfrac12vv_2+\tfrac52v_1^2)+q_0\cos(2w)(-\tfrac32u^2-\tfrac12uv^2\\
&+ \tfrac12u_2+\tfrac12vv_2+v_1^2)+\tfrac12q_0\sin(2w)(-5uv_1-u_1v-v^2v_1+v_3)\\
&+ p_0\cos(2w)(3u^2v+uv^3-\tfrac52uv_2-3u_1v_1-\tfrac32u_2v-\tfrac32v^2v_2-3vv_1^2+\tfrac12v_4)\\
&+ p_0\sin(2w)(3uu_1+6uvv_1+\tfrac32u_1v^2-\tfrac12u_3+v^3v_1-\tfrac32vv_3-\tfrac52v_1v_2).
\end{aligned}
$$

In this extended setting, Eq. (25) acquires a new solution of the form

$$
\begin{pmatrix} F\\ G\end{pmatrix}=\begin{pmatrix}\sum_i(F_i^u p_i+F_i^v q_i+F_i^w r_i)\\ \sum_i(G_i^u p_i+G_i^v q_i+G_i^w r_i)\end{pmatrix},
$$

where

$$
\begin{aligned}
F ={}& (16uu_1+12uvv_1+6u_1v^2-2u_3-6vv_3-5v_1v_2)p_0+(16u^2+12uv^2\\
&- 8u_2-17vv_2-4v_1^2)p_1-3(4u_1+5vv_1)p_2-(8u+5v^2)p_3+p_5\\
&+ (2uv_1+5u_1v-v_3)q_0+(11uv+2v^3-4v_2)q_1-5v_1q_2-4vq_3\\
&+ 2r_3(-2\cos(2w)uv+\cos(2w)v_2-\sin(2w)u_1-2\sin(2w)vv_1)\\
&+ 2r_2(\cos(2w)u_1+2\cos(2w)vv_1-2\sin(2w)uv+\sin(2w)v_2)-3r_1u_1,\\
G ={}& (9uv_1+6u_1v+6v^2v_1-2v_3)p_0+(11uv+2v^3-6v_2)p_1-7v_1p_2-4vp_3\\
&+ (u_1+5vv_1)q_0+(2u+5v^2)q_1-q_3-3v_1r_1+2(-\cos(2w)v_1\\
&+ \sin(2w)u)r_2+2(\cos(2w)u+\sin(2w)v_1)r_3.
\end{aligned}
$$

In the conventional matrix operator form this solution looks as follows:

$$
A' = L+N,
$$

where

$$
L=\begin{pmatrix} L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix},\qquad N=\begin{pmatrix} N_{11} & N_{12}\\ N_{21} & N_{22}\end{pmatrix}
$$

are 2×2-matrix operators corresponding to the local and nonlocal parts of A', respectively, and having the following entries:

$$L_{11} = D_x^5 - (8u + 5v^2)D_x^3 - 3(4u_1 + 5vv_1)D_x^2 + (16u^2 + 12uv^2 - 8u_2 - 17vv_2 - 4v_1^2)D_x + 16uu_1 + 12uvv_1 + 6u_1v^2 - 2u_3 - 6vv_3 - 5v_1v_2,$$
$$L_{12} = -4vD_x^3 - 5v_1D_x^2 + (11uv + 2v^3 - 4v_2)D_x + 2uv_1 + 5u_1v - v_3,$$
$$L_{21} = -4vD_x^3 - 7v_1D_x^2 + (11uv + 2v^3 - 6v_2)D_x + 9uv_1 + 6u_1v + 6v^2v_1 - 2v_3,$$
$$L_{22} = -D_x^3 + (2u + 5v^2)D_x + (u_1 + 5vv_1),$$

and

$$N_{11} = -3Y^u_{1,0}D_x^{-1} \circ Y^u_{1,0} - 4Y^u_{1,1}D_x^{-1} \circ Y^u_{1,1} - 4Y^u_{1,2}D_x^{-1} \circ Y^u_{1,2},$$
$$N_{12} = -3Y^u_{1,0}D_x^{-1} \circ Y^v_{1,0} - 4Y^u_{1,1}D_x^{-1} \circ Y^v_{1,1} - 4Y^u_{1,2}D_x^{-1} \circ Y^v_{1,2},$$
$$N_{21} = -3Y^v_{1,0}D_x^{-1} \circ Y^u_{1,0} - 4Y^v_{1,1}D_x^{-1} \circ Y^u_{1,1} - 4Y^v_{1,2}D_x^{-1} \circ Y^u_{1,2},$$
$$N_{22} = -3Y^v_{1,0}D_x^{-1} \circ Y^v_{1,0} - 4Y^v_{1,1}D_x^{-1} \circ Y^v_{1,1} - 4Y^v_{1,2}D_x^{-1} \circ Y^v_{1,2},$$

whereas

$$Y_{1,0} = \begin{pmatrix} Y^u_{1,0} \\ Y^v_{1,0} \end{pmatrix}, \qquad Y_{1,1} = \begin{pmatrix} Y^u_{1,1} \\ Y^v_{1,1} \end{pmatrix}, \qquad Y_{1,2} = \begin{pmatrix} Y^u_{1,2} \\ Y^v_{1,2} \end{pmatrix}$$

are *symmetries* of the coupled KdV–mKdV system (see [5]) presented in the form

$$Y^u_{1,0} = u_1, \quad Y^v_{1,0} = v_1; \qquad Y^u_{1,1} = -\cos(2w)(\tfrac{1}{2}u_1 + vv_1) + \sin(2w)(uv - \tfrac{1}{2}v_2),$$
$$Y^v_{1,1} = \tfrac{1}{2}\cos(2w)v_1 - \tfrac{1}{2}\sin(2w)u;$$
$$Y^u_{1,2} = \cos(2w)(uv - \tfrac{1}{2}v_2) + \sin(2w)(\tfrac{1}{2}u_1 + vv_1),$$
$$Y^v_{1,2} = -\tfrac{1}{2}\cos(2w)u - \tfrac{1}{2}\sin(2w)v_1.$$

Remark 14. Expressions for the entries of the operator N were obtained as it was indicated in Remark 12.

As above, simulating the techniques developed for the local theory, we have checked that

$$[\![A', A']\!] = 0 \quad \text{and} \quad [\![A', A]\!] = 0,$$

i.e., A and A' are compatible.

Remark 15. Though system (19) is Hamiltonian with respect to A', there does not exist the corresponding Hamiltonian. This can be proved using the same techniques we used in the proof of Theorem 2. Nevertheless, the following facts are valid. Recall that (19) possesses a *recursion operator* [5]. Denote this operator by R and note that our Hamiltonian operators are related to each other by means of this recursion operator, i.e.,

$$A' = R \circ A.$$

Moreover, in the same way one can construct a whole hierarchy of pair-wise compatible Hamiltonian structures. On the other hand, R generates a hierarchy of equations in which (19) is the first term. Then A' is a Hamiltonian structure for all other equations of this hierarchy and these equations possess Hamiltonians with respect to A'.

To make our exposition self-contained, we describe the recursion operator R in the Appendix A.

Acknowledgements

IK and AV are grateful to the University of Twente, where this research was done, for hospitality. The work of AV was supported in part by the NWO and FOM (The Netherlands). We are also grateful to our first reader, S. Igonin, and to the referee for their useful remarks.

Appendix A. Recursion operator for the coupled KdV–mKdV system [5]

To construct the recursion operator, it needs to extend the nonlocal setting introduced above. Namely, we add three new nonlocal variables w_1, w_{11}, and w_{12} defined by

$$
\begin{aligned}
&w_{1,x} = u, \quad w_{1,t} = 3u^2 + 3uv - u_2 - 3vv_2; \qquad w_{11,x} = \cos(2w)w_1 v + \sin(2w)v^2,\\
&w_{11,t} = \cos(2w)(3wuv + wv^3 - wv_2 + uv_1 - u_1 v - v^2 v_1)\\
&\qquad + \sin(2w)(4uv^2 + v^4 - vv_2 - v_1^2); \qquad w_{12,x} = \cos(2w)v^2 - \sin(2w)wv,\\
&w_{12,t} = \cos(2w)(4uv^2 - v^4 - 2vv_2 + v_1^2)\\
&\qquad + \sin(2w)(-3wuv - wv^3 + wv_2 - uv_1 + v^2 v_1)
\end{aligned}
$$

(see Example 5). Then R is a 2×2-matrix operator

$$
R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}
$$

with the entries

$$
\begin{aligned}
R_{11} &= D_x^2 - 4u - v^2 - Y_{1,1}^u D_x^{-1} \circ \psi_{1,2}^u + Y_{1,2}^u D_x^{-1} \circ \psi_{1,1}^u - \tfrac{3}{2} Y_{1,0}^u D_x^{-1},\\
R_{12} &= 2vD_x^2 + v_1 D_x - 3uv + 2v_2 - Y_{1,1}^u D_x^{-1} \circ \psi_{1,2}^v + Y_{1,2}^u D_x^{-1} \circ \psi_{1,1}^v + Y_{2,1}^u D_x^{-1},\\
R_{21} &= -2v - Y_{1,1}^v D_x^{-1} \circ \psi_{1,2}^u + Y_{1,2}^v D_x^{-1} \circ \psi_{1,1}^u - \tfrac{3}{2} Y_{1,0}^v D_x^{-1},\\
R_{22} &= D_x^2 - 2u - v^2 - Y_{1,1}^v D_x^{-1} \circ \psi_{1,2}^v + Y_{1,2}^v D_x^{-1} \circ \psi_{1,1}^v + Y_{2,1}^v D_x^{-1},
\end{aligned}
$$

where $Y_{1,0}$, $Y_{1,1}$, $Y_{1,2}$ are the same symmetries that enter the expression for the nonlocal Hamiltonian structure A', $Y_{2,1}$ is another symmetry with the components

$$
\begin{aligned}
Y_{2,1}^u &= \cos(2w)(-w_{11}u_1 - 2w_{11}vv_1 + 2w_{12}uv - w_{12}v_2) + \sin(2w)\\
&\quad \times (2w_{11}uv - w_{11}v_2 + w_{12}u_1 + 2w_{12}vv_1) - 2uv_1 - 3u_1 v - 2v^2 v_1 + v_3,\\
Y_{2,1}^v &= \cos(2w)(w_{11}v_1 - w_{12}u) - \sin(2w)(w_{11}u + w_{12}v_1) - (u_1 + vv_1),
\end{aligned}
$$

while

$$\psi^u_{1,1} = -\sin(2w), \qquad \psi^v_{1,1} = 2(2\sin(2w)v - w_{12}),$$
$$\psi^u_{1,2} = -\cos(2w), \qquad \psi^v_{1,2} = 2(2\cos(2w)v + w_{11}).$$

Note that $\psi_{1,1} = (\psi^u_{1,1}, \psi^v_{1,1})$ and $\psi_{1,2} = (\psi^u_{1,2}, \psi^v_{1,2})$ are the generating functions for (nonlocal) conservation laws.

References

[1] A.V. Bocharov, V.N. Chetverikov, S.V. Duzhin, N.G. Khor'kova, I.S. Krasil'shchik, A.V. Samokhin, Y.N. Torkhov, A.M. Verbovetsky, A.M. Vinogradov, Symmetries and Conservation Laws for Differential Equations of Mathematical Physics, Monograph, American Mathematical Society, 1999.
[2] S. Igonin, A. Verbovetsky, R. Vitolo, On the formalism of local variational differential operators, Memorandum 1641, Faculty of Mathematical Sciences, University of Twente, The Netherlands, 2002. http://www.math.utwente.nl/publications/2002/1641abs.html.
[3] A.K. Kalkanli, S.Y. Sakovich, I. Yurduşen, Integrability of Kersten–Krasil'shchik coupled KdV–mKdV equations: singularity analysis and Lax pair, J. Math. Phys. 44 (2003) 1703–1708. arXiv:nlin.SI/0206046.
[4] P. Kersten, Supersymmetries and recursion operator for $N = 2$ supersymmetric KdV-equation, Sūrikaisekikenkyūsho Kōkyūroku 1150 (2000) 153–161.
[5] P. Kersten, J. Krasil'shchik, Complete integrability of the coupled KdV–mKdV system, in: T. Morimoto, H. Sato, K. Yamaguchi (Eds.), Lie Groups, Geometric Structures and Differential Equations—One Hundred Years After Sophus Lie, vol. 37 of Advanced Studies in Pure Mathematics, Math. Soc. Jpn., 2002, pp. 151–171.
[6] P. Kersten, I. Krasil'shchik, A. Verbovetsky, On the integrability conditions for some structures related to evolution differential equations, Acta Appl. Math., in press.
[7] I.S. Krasil'shchik, P.H.M. Kersten, Symmetries and Recursion Operators for Classical and Supersymmetric Differential Equations, Kluwer Academic Publishers, Dordrecht, 2000.
[8] J. Krasil'shchik, A.M. Verbovetsky, Homological Methods in Equations of Mathematical Physics, Advanced Texts in Mathematics, Open Education & Sciences, Opava, 1998. arXiv:math.DG/9808130.
[9] B.A. Kupershmidt, Geometry of jet bundles and the structure of Lagrangian and Hamiltonian formalism, in: G. Kaiser, J.E. Marsden (Eds.), Geometric Methods in Mathematical Physics, Lecture Notes in Mathematics, Springer, Berlin, 1980, pp. 162–218.
[10] T. Lada, J. Stasheff, Introduction to sh Lie algebras for physicists, Internat. J. Theoret. Phys. 32 (1993) 1087–1103, arXiv:hep-th/9209099.
[11] P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd ed., Springer, Berlin, 1993.
[12] A.M. Vinogradov, On algebro-geometric foundations of Lagrangian field theory, Sov. Math. Dokl. 18 (1977) 1200–1204.
[13] A.M. Vinogradov, A spectral sequence associated with a nonlinear differential equation and algebro-geometric foundations of Lagrangian field theory with constraints, Sov. Math. Dokl. 19 (1978) 144–148.
[14] A.M. Vinogradov, The $\mathcal{C}$-spectral sequence, Lagrangian formalism, and conservation laws. I. The linear theory. II. The nonlinear theory, J. Math. Anal. Appl. 100 (1984) 1–129.
[15] A.M. Vinogradov, Cohomological Analysis of Partial Differential Equations and Secondary Calculus, American Mathematical Society, 2001.